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ON THE SET OF α, p-BOUNDED VARIATION OF ORDER h

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Abstract. In this paper we first explicit a subset of the set (l_p, l_u) for $1 \leq p < \infty$ and $0 < u < \infty$. Then we deal with the space $bv_p^h(\alpha) = l_p(\alpha)(\Delta^h)$ for $h > 0$ real, generalizing the well-known set of p-bounded variation $bv_p = l_p(\Delta)$, and characterize martix transformations mapping from $bv_p^h(\alpha)$ to $bv_u^k(\beta)$ for $1 \leq p \leq \infty$ and $0 < u \leq \infty$.

1. Preliminaries, background and notation.

Let $A = (a_{nm})_{n,m\geq 1}$ be an infinite matrix and consider the sequence $X = (x_n)_{n\geq 1}$ as a column vector. Then we will define the product $AX = (A_n(X))_{n\geq 1}$ with $A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m$ whenever the series are convergent for all $n \geq 1$. We will denote by s, c_0, c and l_∞ the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. A Banach space E of complex sequences with the norm $\|\|_E$ *is a BK space if each projection* $P_n : X \to P_nX = x_n$ *is continuous. A BK space* E *is said to have AK if every sequence* $X = (x_n)_{n=1}^{\infty} \in E$ *has a unique representation* $X = \sum_{n=1}^{\infty} x_n e_n$ where e_n is the sequence with 1 in the n-th position and 0 otherwise.

For any given subsets E, F of s, we shall say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m\geq 1}$ *maps* E *into* F, that is $A \in (E, F)$, see [4], if

(i) the series defined by $A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all $n \geq 1$ and for all $X \in E$;

(ii) $AX \in F$ for all $X \in E$.

For any subset E of s, we shall write

$$AE = \{Y \in s : Y = AX \text{ for some } X \in E\}.$$

If F is a subset of s, we shall denote the *so-called matrix domain* by

$$F(A) = F_A = \{X \in s : Y = AX \in F\}. \tag{1}$$

In this paper we will consider the well-known set

$$l_p = \left\{X = (x_n)_{n\geq 1} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\} \text{ for } p > 0 \text{ real.}$$

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In the case when $p, u > 0$ are both unequals to 1 except for $p = u = 2$, (see [2]), there is no characterization of the set (l_p, l_u). Denote now

$$U^+ = \{X = (x_n)_{n\geq 1} \in s : x_n > 0 \text{ for all } n\}$$

and let $l_p(\alpha)$ for $\alpha \in U^+$ be the set of all sequences $X = (x_n)_{n\geq 1}$ such that $(x_n/\alpha_n)_{n\geq 1} \in l_p$. The set $l_p(\alpha)$ is a *Banach space with the norm*

$$\|X\|_{l_p(\alpha)} = \left\|D_{\frac{1}{\alpha}} X\right\|_{l_p} = \left[\sum_{n=1}^{\infty}\Big(\frac{|x_n|}{\alpha_n}\Big)^p\right]^{\frac{1}{p}}.$$

Using Wilansky's notation, it can easily be seen that $l_p(\alpha) = (1/\alpha)^{-1} * l_p$ is a *BK space with AK*, see [15, Example 1.13, p.152]. For $p = \infty$ we will write

$$l_\infty(\alpha) = s_\alpha = \Big\{X = (x_n)_{n\geq 1} : \sup_n \frac{|x_n|}{\alpha_n} < \infty\Big\}.$$

For given $\alpha \in U^+$, we also have, see [6, 8, 9, 10]

$$s_\alpha^0 = \Big\{X = (x_n)_{n\geq 1} : \lim_{n\to\infty} \frac{x_n}{\alpha_n} = 0\Big\} \text{ and}$$
$$s_\alpha^{(c)} = \Big\{X = (x_n)_{n\geq 1} : \lim_{n\to\infty} \frac{x_n}{\alpha_n} = l \text{ for some } l \in \mathbb{C}\Big\}.$$

Each of the sets s_α, s_α^0 and $s_\alpha^{(c)}$ is a *BK space* and s_α^0 *has AK*. For α, $\beta = (\beta_n)_{n\geq 1} \in U^+$ we will use the set

$$S_{\alpha,\beta} = \Big\{A = (a_{nm})_{n,m\geq 1} : \sup_n \Big\{\frac{1}{\beta_n}\sum_{m=1}^{\infty} |a_{nm}|\alpha_m\Big\} < \infty\Big\},$$

which is a *Banach space* with the norm $\|A\|_{S_{\alpha,\beta}} = \sup_n\{(1/\beta_n)\sum_{m=1}^{\infty} |a_{nm}|\alpha_m\}$, see [5-12]. If $s_\alpha = s_\beta$ we get the *Banach algebra with identity* $S_{\alpha,\alpha} = S_\alpha$, see [5, 8, 11].

We will use the operator Δ defined by $\Delta x_1 = x_1$ and $\Delta x_n = x_n - x_{n-1}$ for $n \geq 2$ and for all $X = (x_n)_{n\geq 1}$ and define the set of *α, p-bounded variation of order* 1, by

$$bv_p(\alpha) = \Big\{X = (x_n)_{n\geq 1} : \sum_{n=1}^{\infty}\Big(\frac{|x_n - x_{n-1}|}{\alpha_n}\Big)^p < \infty\Big\}, \quad \text{with } x_0 = 0.$$

Recall that for $\alpha = e = (1, \ldots, 1, \ldots)$, we have $bv_p(\alpha) = bv_p$ and bv_p is the *set of p-bounded variation*, and for $p = 1$ and $p = \infty$, the space bv_p is reduced to the spaces bv and $l_\infty(\Delta)$ respectively. Using the notation (1) we may redefine the space $bv_p(\alpha)$ as

$$bv_p(\alpha) = l_p(\alpha)(\Delta).$$

There are some results on the sets (bv_p, Y) with $Y = l_\infty$, c_0, c, l_1, or bv in [1, Theorem 13.3 and Theorem 13.4, pp.52]. When p is replaced by a sequence $\widetilde{p} = (p_n)_{n\geq 1}$ there are other results on $(bv_{\widetilde{p}}, Y)$ where Y is either of the sets l_∞, c_0, c, l_1, see [3, Theorem 3.2, pp.160]. Here we give *conditions for a matrix map to belong to* $(bv_p^h(\alpha), bv_u^k(\beta))$ where h, $k > 0$, $1 \le p \le \infty$, $0 < u < \infty$, and $bv_p^h(\alpha) = l_p(\alpha)(\Delta^h)$.

2. Subset of (l_p, l_u) with $1 \le p < \infty$ and $0 < u < \infty$

Let p, u be reals with $p \ge 1$ and $u > 0$. For any given infinite matrix A, put

$$N_{p,u}(A) = \begin{cases} \sup_{m\ge 1}\left(\sum_{n=1}^{\infty} |a_{nm}|\right) & \text{if } u = p = 1, \\ \left[\sum_{n=1}^{\infty}\left(\sup_{m\ge 1} |a_{nm}|\right)^u\right]^{\frac{1}{u}} & \text{if } p = 1 \text{ and } 0 < u < \infty, u \ne 1; \\ \left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} |a_{nm}|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} & \text{if } 1 < p < \infty, 0 < u < \infty \text{ with } q = p/(p-1). \end{cases}$$

We will write $L_{p,u}$ for the set of all infinite matrices A with $N_{p,u}(A) < \infty$. We then have the following result

Theorem 1. *Let p, u be reals with $p \ge 1$ and $u > 0$. Then*

$$L_{p,u} \subset (l_p, l_u)$$

and for any given $A \in L_{p,u}$, $\|AX\|_{l_u} \le N_{p,u}(A)\|X\|_{l_p}$ for all $X \in l_p$.

Proof. Case $u = p = 1$. We have $A \in (l_1, l_1)$ if and only if all the series $\sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all n for all $X \in l_1$ and $AX \in l_1$ for all $X \in l_1$. Let $A \in L_{1,1}$ we get

$$\begin{aligned} \|AX\|_{l_1} &\le \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} |a_{nm}x_m|\right) \\ &\le \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} |a_{nm}x_m|\right) \\ &\le \left(\sum_{m=1}^{\infty} |x_m|\right)\left(\sup_{m\ge 1}\sum_{n=1}^{\infty} |a_{nm}|\right) = \|A^t\|\,\|X\|_{l_1} \text{ for all } X \in l_1. \end{aligned}$$

Case $p = 1$ and $u > 0$, $u \ne 1$. As above, let $A \in L_{1,u}$. For every $X \in l_1$ we successively get

$$\begin{aligned} \|AX\|_{l_u}^u &\le \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} |a_{nm}x_m|\right)^u \\ &\le \sum_{n=1}^{\infty}\left[\left(\sup_{m\ge 1} |a_{nm}|\right)\sum_{m=1}^{\infty} |x_m|\right]^u \\ &\le \sum_{n=1}^{\infty}\left(\sup_{m\ge 1} |a_{nm}|\right)^u\left(\sum_{m=1}^{\infty} |x_m|\right)^u. \end{aligned}$$

We conclude

$$\|AX\|_{l_u} \leq \left[\sum_{n=1}^{\infty}\left(\sup_{m\geq 1}|a_{nm}|\right)^u\right]^{\frac{1}{u}} \qquad \|X\|_{l_1} = [N_{1,u}(A)]\,\|X\|_{l_1}.$$

Case $p > 1$ and $u > 0$. Let $A \in L_{p,u}$. For every $X \in l_p$, we get

$$\|AX\|_{l_u}^u = \sum_{n=1}^{\infty}\left(\left|\sum_{m=1}^{\infty} a_{nm}x_m\right|^u\right) \leq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}x_m|\right)^u$$

and by the Hölder inequality, where $q = p/(p-1)$, we have

$$\begin{aligned}\|AX\|_{l_u}^u &\leq \sum_{n=1}^{\infty}\left[\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{\frac{1}{q}}\left(\sum_{m=1}^{\infty}|x_m|^p\right)^{\frac{1}{p}}\right]^u \\ &\leq \sum_{n=1}^{\infty}\left[\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{\frac{1}{q}}\|X\|_{l_p}\right]^u \\ &\leq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{\frac{u}{q}}\|X\|_{l_p}^u \leq [N_{p,u}(A)]^u\|X\|_{l_p}^u.\end{aligned}$$

Remark 1. Let us recall the next results due to Stieglitz and Tietz [16], and Maddox [4], where either p or u is equal to one:

$$(l_1, l_u) = \left\{A = (a_{nm})_{n,m\geq 1} : \sup_{m\geq 1}\left(\sum_{n=1}^{\infty}|a_{nm}|^u\right) < \infty\right\} \quad for\, 1 \leq u < \infty,$$

and if $1 < p < \infty$ and $q = p/(p-1)$, then

$$(l_p, l_1) = \left\{A = (a_{nm})_{n,m\geq 1} : \sup_{N\subset\mathbb{N}, N \text{ finite}}\left(\sum_{m=1}^{\infty}\left|\sum_{n\in N} a_{nm}\right|^q\right) < \infty\right\}.$$

We can also remark that if $u = p \geq 1$, then $\|A\|_{(l_p,l_p)} \leq N_{p,p}(A)$ with

$$N_{p,p}(A) = \begin{cases} \|A^t\|_{S_1} & \text{for } p = 1, \\ \left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} & \text{for } p > 1. \end{cases}$$

We have the following application.

Example 2. Let θ, $u > 0$ and $p > 1$ be reals and consider the triangle

$$C^{\theta} = \begin{pmatrix} 1 & & & & \\ . & . & & \mathrm{O} & \\ \frac{1}{n^{\theta}} & . & \frac{1}{n^{\theta}} & & \\ . & . & . & & . \end{pmatrix}.$$

Then $C^\theta \in (l_p, l_u)$ for $\theta > 1/u + 1/q$ with $q = p/(p-1)$.

Proof. Let $f(x) = x^\theta$. Since $n/f^q(n)$ is decreasing sequence, writing $C^\theta \in (a_{nm})_{n,m\geq 1}$ we have

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|a_{nm}|^q\right)^{\frac{u}{q}} = \sum_{n=1}^{\infty}\left(\frac{n}{f^q(n)}\right)^{\frac{u}{q}} \leq \int_1^\infty \left(\frac{x}{f^q(x)}\right)^{\frac{u}{q}} dx.$$

Now $(x/f^q(x))^{u/q} = 1/x^{(q^{\theta-1})u/q}$ and $\int_1^\infty [x/(f^q(x))]^{u/q} dx < \infty$ for $(q^{\theta-1})u/q > 1$, that is $\theta > 1/u + 1/q$.

3. Some properties of the set $bv_p^h(\alpha)$.

First recall some well known properties of the sets bv and $bv^0 = bv \bigcap c_0$. In the following $T = (t_{nm})_{n,m\geq 1}$ is a triangle if $t_{nm} = 0$ for all $m > n$ and $t_{nn} \neq 0$ for all n.

Theorem 3.([15, Theorems 3.3, 3.5, pp. 178, 179], [17, Theorems 4.3.12, 4.3.14, pp. 63, 64]).
Let E be a BK space. Then E_T is a BK space with $\|X\|_T = \|TX\|_E$.
If E is a closed subset of F then E_T is a closed subspace of F_T.

The set $bv = l_1(\Delta)$ is called the *set of bounded variation* and by Theorem 3 and [14, Theorem 2.2.10, p.152] if we put $bv^0 = bv \bigcap c_0$, then bv^0 and bv are *BK spaces with their natural norm* $\|X\|_{bv} = \sum_{n=1}^\infty |x_n - x_{n-1}|$. *The set bv^0 has AK and every sequence $X = (x_n)_{n\geq 1} \in bv$ has a unique representation $X = le + \sum_{n=1}^\infty (x_n - l)e_n$ where* $l = \lim_{n\to\infty} x_n$.

Here for $\alpha \in U^+$ we define the set of α, *p-bounded variation of order h*, by $bv_p^h(\alpha) = l_p(\alpha)(\Delta^h)$ for $0 < p \leq \infty$ and $h > 0$. We will put $bv_p^1(\alpha) = bv_p(\alpha)$, $bv^h(\alpha) = l_1(\alpha)(\Delta^h)$ and for $p = \infty$, it can easily be seen that $bv_\infty^h(\alpha) = s_\alpha(\Delta^h)$.

We need to recall some results given in [8]. For this consider the following sets

$$\widehat{C_1} = \left\{X = (x_n)_{n\geq 1} \in U^+ : \frac{1}{x_n}\left(\sum_{k=1}^n x_k\right) = O(1)\ (n\to\infty)\right\},$$

$$\widehat{C_1^+} = \left\{X \in U^+ \bigcap cs : \frac{1}{x_n}\left(\sum_{k=1}^n x_k\right) = O(1)\ (n\to\infty)\right\},$$

$$\Gamma = \left\{X \in U^+ : \overline{\lim_{n\to\infty}}\left(\frac{x_{n-1}}{x_n}\right) < 1\right\},$$

$$\hat{\Gamma} = \left\{X \in U^+ : \lim_{n\to\infty}\left(\frac{x_{n-1}}{x_n}\right) < 1\right\},$$

$$\Gamma^+ = \left\{X \in U^+ : \overline{\lim_{n\to\infty}}\left(\frac{x_{n+1}}{x_n}\right) < 1\right\}.$$

Note that $X \in \Gamma^+$ if and only if $1/X \in \Gamma$. We shall see in Lemma 4 that if $X \in \widehat{C_1}$, then $x_n \to \infty (n\to\infty)$. Furthermore, $X \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(X) = \sup_{n \geq q+1} \left(\frac{x_{n-1}}{x_n}\right) < 1.$$

We obtain the following results in which we put

$$[C(X)X]_n = \frac{1}{x_n}\left(\sum_{k=1}^{n} x_k\right).$$

Lemma 4. *Let $\alpha \in U^+$.*

(i) *If $\alpha \in \widehat{C_1}$ there are $K > 0$ and $\gamma > 1$ such that $\alpha_n \geq K\gamma^n$ for all n.*

(ii) *The condition $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C_1}$ and there exists a real $b > 0$ such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\gamma_q(\alpha)} + b[\gamma_q(\alpha)]^n \quad for\, n \geq q+1.$$

(iii) *The condition $\alpha \in \Gamma^+$ implies $\alpha \in \widehat{C_1^+}$.*

The proof follows from [9, Proposition 2.1, p. 1656-1658].

Remark 2. Note that $\Gamma \not\subseteq \widehat{C_1}$.

Let us consider now Δ as an operator from E into itself where E is either of the sets s_α, s_α^0, $s_\alpha^{(c)}$, or $l_p(\alpha)$. Then we obtain *conditions for $\Delta \in (E,E)$ to be bijective.* In this way we have the following results.

Lemma 5. *Let $\alpha \in U^+$.*

(i) *If $\alpha \in \Gamma$ then $bv_p(\alpha) = l_p(\alpha)$ for $1 \leq p \leq \infty$;*

(ii) *$s_\alpha(\Delta) = s_\alpha$ if and only if $\alpha \in \widehat{C_1}$;*

(iii) *$s_\alpha^0(\Delta) = s_\alpha^0$ if and only if $\alpha \in \widehat{C_1}$;*

(iv) *$s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ if and only if $\alpha \in \widehat{\Gamma}$;*

(v) *$\Delta_\alpha = D_{\frac{1}{\alpha}}\Delta D_\alpha$ is bijective from c into itself with $\lim X = \Delta_\alpha - \lim X$, if and only if*

$$\frac{\alpha_{n-1}}{\alpha_n} \to 0.$$

Proof. (i) comes from [10]. (ii), (iii) and (v) come from [8, Theorem 2.6, pp. 1789] and (iv) is a direct consequence of [8, Theorem 2.6, pp. 1789] and [12, Proposition 2, pp. 88].

Remark 3. Note that by Lemma 4(ii) the condition $\alpha \in \Gamma$ implies $s_\alpha(\Delta) = s_\alpha$ and $s_\alpha^0(\Delta) = s_\alpha^0$.

For $h \in \mathbb{R}$ put now

$$\binom{-h+i-1}{i} = \begin{cases} \frac{-h(-h+1)\cdots(-h+i-1)}{i!} & \text{if } i > 0, \\ 1 & \text{if } i = 0, \end{cases}$$

and define the operator $\Delta^h = (\tau_{nm})_{n,m\geq 1}$ for $h \in \mathbb{R}$ by

$$\tau_{nm} = \begin{cases} \begin{pmatrix} -h+n-m-1 \\ n-m \end{pmatrix} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For $h = -1$ we get $\Delta^h = \Sigma$ with $\Sigma_{nm} = 1$ if $m \leq n$ and $\Sigma_{nm} = 0$ for $m > n$, see [5]. Study now the identity $bv_p^h(\alpha) = l_p(\alpha)(\Delta^h) = l_p(\alpha)$ for $h > 0$ or $h \geq 1$ integer and $1 \leq p < \infty$.

We obtain the following

Lemma 6.([10]) *Let $\alpha \in U^+$.*

(i) *For any given real $h > 0$, the condition $bv^h(\alpha) = l_1(\alpha)$ is equivalent to*

$$\alpha_n \left(\sum_{m=n}^{\infty} \begin{pmatrix} h+m-n-1 \\ m-n \end{pmatrix} \frac{1}{\alpha_m} \right) = O(1)(n \to \infty);$$

(ii) *Let $h \geq 1$ be an integer and $p \geq 1$ a real. If $\alpha \in \Gamma$ then*

$$bv_p^h(\alpha) = l_p(\alpha).$$

Remark 4. Note that we also have $1/\alpha \in \widehat{C_1^+}$ if and only if $bv(\alpha) = l_1(\alpha)$. Indeed the conditions $\Delta \in (l_1(\alpha), l_1(\alpha))$ and $\Sigma \in (l_1(\alpha), l_1(\alpha))$ are equivalent to $\Delta^+ \in S_{1/\alpha}$ and $\Sigma^+ \in S_{1/\alpha}$, that is

$$\frac{\alpha_n}{\alpha_{n-1}} = O(1) \quad \text{and} \quad \alpha_n \left(\sum_{k=1}^{n} \frac{1}{\alpha_k} \right) = O(1)(n \to \infty).$$

From the inequality $\alpha_n/\alpha_{n-1} \leq \alpha_n \left(\sum_{k=1}^{n} 1/\alpha_k \right)$ for all n, we conclude that $1/\alpha \in \widehat{C_1^+}$ if and only if $bv(\alpha) = l_1(\alpha)$.

4. Matrix map from $bv_p^h(\alpha)$ to $bv_u^k(\beta)$

In this section we give necessary conditions for an infinite matrix A to map $bv_p^h(\alpha) = l_p(\alpha)(\Delta^h)$ into $bv_u^k(\beta)$ and some characterizations of the sets $(bv^h(\alpha), bv_u^k(\beta))$, $(bv_p^h(\alpha), bv_\infty^k(\beta))$ and $(bv_\infty^h(\alpha), bv_\infty^k(\beta))$. For this we need additional results.

4.1. Other results

To state the next results we first need to recall the characterizations of (l_p, l_∞) and (l_∞, l_u) and consider the identity $A(\chi X) = (A\chi)X$ for $X \in E$, where E is either of the sets $l_p(\alpha)$, $1 \leq p \leq \infty$, s_α, or s_α^0. In this way we have, (see [15] and [16]).

Lemma 7.
(i) *$A \in (l_p, l_\infty)$ if and only if*

$$\begin{cases} \sup_{n,m} |a_{nm}| < \infty & \text{for } p = 1, \\ \sup_n \sum_{m=1}^{\infty} |a_{nm}|^q < \infty & \text{for } 1 < p < \infty \text{ and } q = \frac{p}{(p-1)}. \end{cases}$$

(ii) *Let $1 \leq u < \infty$. Then $A \in (l_\infty, l_u)$ if and only if*

$$\sup_{N \subset \mathbb{N}, N \text{ finite}} \left(\sum_{n=1}^{\infty} \left| \sum_{m \in N} a_{nm} \right|^u \right) < \infty.$$

We also need the following lemmas.

Lemma 8. *Let $p > 1$ be a real and $\chi = (\chi_{nm})_{n,m \geq 1}$ an infinite matrix. The identity $A(\chi X) = (A\chi)X$ for all $X \in E$ holds in the following cases*

(i) *When $E = l_1(\alpha)$ if*

$$\sum_{m=1}^{\infty} |a_{nm}| < \infty \qquad \text{for all } n, \text{ and } \sup_{n,m}(|\chi_{nm}| \alpha_m) < \infty; \tag{2}$$

(ii) *When $E = l_p(\alpha)$ with $1 < p < \infty$ if*

$$\sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{m=1}^{\infty} |\chi_{km}|^q \alpha_m^q \right)^{\frac{1}{q}} < \infty \qquad \text{for all } n, \text{ with } q = \frac{p}{p-1}; \tag{3}$$

(iii) *When $E \in \left\{ s_\alpha, s_\alpha^0, s_\alpha^{(c)} \right\}$ if*

$$\sum_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} |a_{nk} \chi_{km}| \alpha_m \right) < \infty \qquad \text{for all } n. \tag{4}$$

Proof. First note that for any given integer n, we have

$$A_n(\chi X) = \sum_{k=1}^{\infty} a_{nk} \left(\sum_{m=1}^{\infty} \chi_{km} x_m \right) \quad \text{for } X = (x_n)_{n \geq 1} \in s,$$

whenever the series in the second member are convergent.

(i) Assume that (2) holds. Then putting

$$|A_n|(|\chi X|) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |a_{nk}| \, |\chi_{km}| \, |x_m| \text{ for } n \geq 1$$

one gets

$$|A_n|(|\chi X|) \leq \sum_{k=1}^{\infty} |a_{nk}| \sup_{m,k}(|\chi_{km}|\alpha_m) \sum_{m=1}^{\infty} \frac{|x_m|}{\alpha_m}$$
$$\leq \sum_{k=1}^{\infty} |a_{nk}| \sup_{m,k}(|\chi_{km}|\alpha_m) \|X\|_{l_1(\alpha)} < \infty \text{ for all } n \text{ and all } X \in l_1(\alpha).$$

So we can invert $\sum_k$ and $\sum_m$ in the expression of y_n. This shows $A(\chi X) = (A\chi)X$ for all $X \in l_1(\alpha)$.

(ii) Assume that (3) holds. Then by the Hölder inequality

$$|A_n|(|\chi X|) = \sum_{k=1}^{\infty} \left(|a_{nk}| \sum_{m=1}^{\infty} \left(|\chi_{km}| \alpha_m \frac{|x_m|}{\alpha_m} \right) \right)$$
$$\leq \sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{m=1}^{\infty} |\chi_{km}|^q \alpha_m^q \right)^{\frac{1}{q}} \left(\sum_{m=1}^{\infty} \left(\frac{|x_m|}{\alpha_m} \right)^p \right)^{\frac{1}{p}}$$
$$\leq \sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{m=1}^{\infty} |\chi_{km}|^q \alpha_m^q \right)^{\frac{1}{q}} \|X\|_{l_p(\alpha)} \text{ for all } n \text{ and for all } X \in l_p(\alpha);$$

and we conclude reasoning as above.

(iii) Comes from the fact that if (4) holds then $|A_n|(|\chi X|) < \infty$ for all n and all $X \in s_\alpha$. We get the same result when s_α is replaced by s_α^0 and by $s_\alpha^{(c)}$, since these spaces are included in s_α. This completes the proof.

We also need to recall the following well-known result given in [13, Theorem 1].

Lemma 9. *Let $T \in £$. Then for arbitrary subsets E and F of s, the condition $A \in (E, F(T))$ is equivalent to $TA \in (E, F)$.*

4.2. Properties of the set $(bv_p^h(\alpha), bv_u^k(\beta))$ for $1 < p < \infty$, $0 < u < \infty$, h and k being reals or integers

First we give necessary conditions to have $A \in (bv_p^h(\alpha), bv_u^k(\beta))$, this gives the following

Theorem 10. *Let $\alpha, \beta \in U^+$ and $1 < p < \infty$.*

(i) *Let $0 < u < \infty$.*

(a) *Let $k \in \mathbb{R}$ and $h \geq 1$ be an integer. If $\alpha \in \Gamma$, the condition*

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \left(\left| \frac{1}{\beta_n} \sum_{j=1}^{n} \binom{-k+n-j-1}{n-j} a_{jm} \right| \alpha_m \right)^q \right)^{\frac{u}{q}} < \infty \quad \text{with } q = \frac{p}{p-1}, \tag{5}$$

implies $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

(b) *Let* $\beta \in \Gamma$, $h \in \mathbb{R}$ *and* $k \geq 1$ *be an integer. Assume*

$$\sum_{m=1}^{\infty} |a_{nm}| \left(\sum_{j=m}^{\infty} \left| \binom{h+j-m-1}{j-m} \right|^q \alpha_m^q \right) < \infty \quad \textit{for all } n \tag{6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^u} \left(\sum_{m=1}^{\infty} \left| \sum_{j=m}^{\infty} a_{nj} \binom{h+j-m-1}{j-m} \alpha_m \right|^q \right)^{\frac{u}{q}} < \infty, \tag{7}$$

then $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

(ii) *Let* $1 \leq u < \infty$ *and* $h, k \geq 1$ *integers. If* $\alpha, \beta \in \Gamma$, *then condition*

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n^u} \left(\sum_{m=1}^{\infty} (|a_{nm}| \alpha_m)^q \right)^{\frac{u}{q}} < \infty$$

implies $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

Proof. First $\alpha \in \Gamma$ implies Δ is bijective from $l_p(\alpha)$ into itself and

$$bv_p^h(\alpha) = l_p(\alpha)(\Delta^h) = l_p(\alpha).$$

Then $A \in (bv_p^h(\alpha), bv_u^k(\beta))$ if and only if $D_{1/\beta}\Delta^k A \in (bv_p^h(\alpha), l_u) = (l_p(\alpha), l_u)$; and $D_{1/\beta}\Delta^k A D_\alpha \in (l_p, l_u)$ if $D_{1/\beta}\Delta^k A D_\alpha \in L_{p,u}$. We have

$$D_{\frac{1}{\beta}} \Delta^k A D_\alpha = \left(\frac{1}{\beta_n} \left(\sum_{j=1}^{n} \binom{-k+n-j-1}{n-j} a_{jm} \right) \alpha_m \right)_{n,m \geq 1},$$

and using Lemma 8(ii), we conclude that condition (5) implies $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

(i)(b) Since $\beta \in \Gamma$, then Δ is bijective from $l_u(\beta)$ to itself and it is the same for Δ^k. So

$$bv_u^k(\beta) = l_u(\beta)(\Delta^k) = l_u(\beta).$$

We have $\Delta^{-h} = (\tau_{nm})_{n,m \geq 1}$ with

$$\tau_{nm} = \begin{cases} \binom{h+n-m-1}{n-m} & \text{for } m \leq n, \\ 0 & \text{for } m > n. \end{cases}$$

By Lemma 8, condition (6) permits us to write that

$$A(\Delta^{-h} X) = (A\Delta^{-h})X \qquad \text{for all } X \in l_p(\alpha). \tag{8}$$

Now since $A\Delta^{-h} = (c_{nm})_{n,m \geq 1}$ with

$$c_{nm} = \sum_{j=m}^{\infty} a_{nj} \binom{h+j-m-1}{j-m}$$

condition (7) means that $D_{1/\beta}A\Delta^{-h}D_\alpha \in L_{p,u}$; and since $L_{p,u} \subset (l_p, l_u)$ then $A\Delta^{-h} \in (l_p(\alpha), l_u(\beta))$. Thus $(A\Delta^{-h})X \in l_u(\beta)$ for all $X \in l_p(\alpha)$ and (8) implies that the series defined by $A_n(\Delta^{-h}X)$ are convergent for all n and for all $X \in l_p(\alpha)$, and $A(\Delta^{-h}X) \in l_u(\beta)$. We conclude that $D_{1/\beta}A\Delta^{-h}D_\alpha \in L_{p,u}$ implies $A \in (bv_p^h(\alpha), l_u(\beta))$ and $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

Statement (ii) The condition $\alpha, \beta \in \Gamma$ implies $bv_p^h(\alpha) = l_p(\alpha)$ and $bv_u^k(\beta) = l_u(\beta)(\Delta^k) = l_u(\beta)$. Then $D_{1/\beta}AD_\alpha = (a_{nm}\alpha_m/\beta_n)_{n,m\geq 1} \in L_{p,u}$ implies $A \in (bv_p^h(\alpha), bv_u^k(\beta))$.

Until now were given necessary conditions for A to belong to $(bv_p^h(\alpha), bv_u(\beta))$, when $u = 1$ and $h \in \mathbb{R}$ we get the next characterization. In all that follows we will need to use the convention $a_{0m} = 0$ for all m.

Proposition 11. *Let $1 < p < \infty$, h be a real and assume that*

$$\sum_{j=1}^{\infty} |a_{nj} - a_{n-1,j}| \left(\sum_{m=1}^{\infty} \left| \binom{h+j-m-1}{j-m} \alpha_m \right|^q \right) < \infty \quad \textit{for all } n. \tag{9}$$

Then $A \in (bv_p^h(\alpha), bv(\beta))$ if and only if

$$\sup_{N\subset\mathbb{N}, N \textit{ finite}} \sum_{m=1}^{\infty} \frac{1}{\beta_n} \left| \sum_{n\in N} \sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \binom{h+j-m-1}{j-m} \alpha_m \right|^q < \infty.$$

Proof. First $A \in (bv_p^h(\alpha), bv(\beta))$ if and only if

$$\Delta A(\Delta^{-h}X) \in l_1(\beta) \qquad \text{for all } X \in l_p(\alpha).$$

Now since $\Delta A = (a_{nm} - a_{n-1,m})_{n,m\geq 1}$, from Lemma 8(ii), we see that under condition (9) $\Delta A(\Delta^{-h}X) = (\Delta A\Delta^{-h})X$ for all $X \in l_p(\alpha)$. Then $A \in (bv_p^h(\alpha), bv(\beta))$ if and only if $\Delta A\Delta^{-h} \in (l_p(\alpha), l_1(\beta))$ and we conclude using the characterization of (l_p, l_1).

Remark 5. Note that for $h, k \in \mathbb{R}$, we have $A \in (bv_p^h(\alpha), bv_u^k(\beta))$ if $D_{1/\beta}\Delta^k A\Delta^{-h}D_\alpha \in L_{p,u}$ when the identity

$$\Delta^k A(\Delta^{-h}X) = (\Delta^k A\Delta^{-h})X \qquad \textit{for all } X \in l_p(\alpha)$$

is satisfied.

4.3. Properties of the set $(bv^h(\alpha), bv_u^k(\beta))$ for $h > 0$ and k real or integer

Now we can state the next results

Theorem 12. *Let $1 \leq u < \infty$ and $h > 0$.*

(i) *Assume $\alpha \in \Gamma$ and $k \in \mathbb{R}$. Then $A \in (bv^h(\alpha), bv_u^k(\beta))$ if and only if*

$$\sup_m \sum_{n=1}^{\infty} \frac{1}{\beta_n^u} \left(\left| \sum_{j=1}^{n} a_{jm} \binom{-k+n-j-1}{n-j} \right| \alpha_m \right)^u < \infty; \tag{10}$$

(ii) *Let $\beta \in \Gamma$ and $k \geq 1$ be an integer. Under the condition*

$$\sum_{m=1}^{\infty} |a_{nm}| < \infty \quad \textit{for all } n, \textit{ and } \sup_{n,m} \Big\{ \Big| \binom{h+n-m-1}{n-m} \Big| \alpha_m \Big\} < \infty, \tag{11}$$

$A \in (bv^h(\alpha), bv_u^k(\beta))$ if and only if

$$\sup_m \sum_{n=1}^{\infty} \frac{1}{\beta_n^u} \Big| \sum_{j=m}^{\infty} a_{nj} \binom{h+j-m-1}{j-m} \alpha_m \Big|^u < \infty.$$

(iii) *Let $\alpha, \beta \in \Gamma$ and $k \geq 1$ be integer. Then $A \in (bv^h(\alpha), bv_u^k(\beta))$ if and only if*

$$\sup_m \Big\{ \alpha_m^u \sum_{n=1}^{\infty} \Big(\frac{|a_{nm}|}{\beta_n} \Big)^u \Big\} < \infty. \tag{12}$$

Proof. (i) The condition $\alpha \in \Gamma$ implies $bv^h(\alpha) = l_1(\alpha)$. So $A \in (bv^h(\alpha), bv_u^k(\beta))$ if and only if $\Delta^k A \in (l_1(\alpha), l_u(\beta))$. From the expression of $D_{1/\beta}\Delta^k A D_\alpha$ in the proof of Theorem 10(i)(a), we conclude that $D_{1/\beta}\Delta^k A D_\alpha \in (l_1, l_u)$ if and only if (10) holds.

(ii) The condition $A \in (bv^h(\alpha), l_u(\beta))$ means that the series defined by $A_n(\Delta^{-h}X)$ are convergent for all $X \in l_1(\alpha)$ and for all n and

$$A(\Delta^{-h}X) \in l_u(\beta) \qquad \text{for all } X \in l_1(\alpha).$$

Under condition (11), $A(\Delta^{-h}X) = (A\Delta^{-h})X$ for all $X \in l_1(\alpha)$, so $A \in (bv^h(\alpha), l_u(\beta))$ if and only if $D_{1/\beta}A\Delta^{-h}D_\alpha \in (l_1, l_u)$, and we conclude since $\beta \in \Gamma$ implies $bv_u(\beta) = l_u(\beta)$.

(iii) Here $\alpha, \beta \in \Gamma$ implies $bv^h(\alpha) = l_1(\alpha)(\Delta^h) = l_1(\alpha)$ and $bv_u^k(\beta) = l_u(\beta)$. So $A \in (l_1(\alpha), l_u(\beta))$ if and only if $D_{1/\beta}AD_\alpha \in (l_1, l_u)$ and we conclude using the characterization of (l_1, l_u).

Remark 6. We also have the next result. Let $k \in \mathbb{R}$, $1 \leq u < \infty$ and $\alpha \in l_\infty$. Then under the condition

$$\sum_{m=1}^{\infty} |a_{nm} - a_{n-1,m}| < \infty \qquad \textit{for all } n, \tag{13}$$

we have $A \in (bv(\alpha), bv_u^k(\beta))$ if and only

$$\sup_m \Big\{ \sum_{n=1}^{\infty} \frac{1}{\beta_n^u} \Big| \sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \binom{h+j-m-1}{j-m} \alpha_m \Big|^u \Big\} < \infty.$$

Indeed $A \in (bv(\alpha), bv_u^k(\beta))$ if only if $\Delta A(\Delta^{-h}X) \in l_u(\beta)$ for all $X \in l_1(\alpha)$. Since $\alpha \in l_\infty$ and (13) holds, by Lemma 8(i) we have $\Delta A(\Delta^{-h}X) = (\Delta A(\Delta^{-h})X$ for all $X \in l_1(\alpha)$. We conclude since $\Delta A\Delta^{-h} \in (l_1(\alpha), l_u(\beta))$ and

$$\Delta A\Delta^{-h} = \Big(\sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \binom{h+j-m-1}{j-m} \Big)_{n,m \geq 1}.$$

Remark 7. Note that (ii) in the previous theorem is true for h real.

4.4. The sets $(bv_p(\alpha), bv_\infty(\beta))$ and $(bv_\infty(\alpha), bv_u(\beta))$

In this part we characterize the set $(bv_p(\alpha), bv_\infty(\beta))$ in the cases when $1 \le p < \infty$, $u = \infty$ and $p = \infty$, $1 \le u < \infty$. Then we get the following result.

Theorem 13. *Let $\alpha \in U^+$.*

(i) *Assume*

$$\sum_{m=1}^{\infty} |a_{nm} - a_{n-1,m}| < \infty \qquad \text{for all } n \ge 1 \text{ and } \alpha \in l_\infty. \tag{14}$$

Then $A \in (bv(\alpha), bv_\infty(\beta))$ if and only if

$$\sup_{n,m} \frac{1}{\beta_n} \left| \sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \right| \alpha_m < \infty. \tag{15}$$

(ii) *Let $1 < p < \infty$.*

(a) *Under the condition*

$$\sum_{k=1}^{\infty} |a_{nk} - a_{n-1,k}| \left(\sum_{m=k}^{\infty} \alpha_m^q \right)^{\frac{1}{q}} < \infty \qquad \text{for all } n \ (\text{with } q = \frac{p}{p-1}), \tag{16}$$

we have $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if

$$\sup_n \frac{1}{\beta_n^q} \sum_{m=1}^{\infty} \left| \alpha_m \sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \right|^q < \infty. \tag{17}$$

(b) *If $\beta \in \Gamma$, under the condition*

$$\sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{m=k}^{\infty} \alpha_m^q \right)^{\frac{1}{q}} < \infty \qquad \text{for all } n \ (\text{with } q = \frac{p}{p-1}), \tag{18}$$

$A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if

$$\begin{cases} \sup_{n,m} \left(\frac{1}{\beta_n} \sum_{j=m}^{\infty} |a_{nj}| \alpha_m \right) < \infty & \text{for } p = 1, \\ \sup_n \left[\frac{1}{\beta_n^q} \sum_{m=1}^{\infty} \left| \sum_{j=m}^{\infty} a_{nj} \alpha_m \right|^q \right] < \infty & \text{for } 1 < p < \infty. \end{cases}$$

(iii) *Under the condition*

$$\sum_{m=1}^{\infty} \alpha_m \sum_{j=m}^{\infty} |a_{nj}| < \infty \qquad \text{for all } n, \tag{19}$$

$A \in (bv_\infty(\alpha), bv_\infty(\beta))$ if and only if

$$\sup_n \frac{1}{\beta_n} \sum_{m=1}^{\infty} \alpha_m \left| \sum_{j=m}^{\infty} (a_{nj} - a_{n-1,j}) \right| < \infty. \tag{20}$$

Proof. Since $bv_\infty(\beta) = s_\beta(\Delta)$, we have $A \in (bv(\alpha), bv_\infty(\beta))$ if and only if $\Delta A \in (bv(\alpha), s_\beta)$. Then from the identity $bv(\alpha) = l_1(\alpha)(\Delta)$, we have $\Delta A \in (bv(\alpha), s_\beta)$ if and only if

$$(\Delta A)(\Sigma X) \in s_\beta \text{ for all } X \in l_1(\alpha);$$

and by Lemma 8(i), the conditons given by (14) imply $(\Delta A)(\Sigma X) = (\Delta A \Sigma)X$ for all $X \in l_1(\alpha)$. Now we successively get $A\Sigma = \left(\sum_{k=m}^{\infty} a_{nk}\right)_{n,m \geq 1}$ and $\Delta A \Sigma = \left(\sum_{k=m}^{\infty} (a_{nk} - a_{n-1,k})\right)_{n,m \geq 1}$ and we conclude that $A \in (bv(\alpha), bv_\infty(\beta))$ if and only if $D_{1/\beta} \Delta A \Sigma D_\alpha \in (l_1, l_\infty)$, that is condition (15).

(ii)(a) Since $bv_\infty(\beta) = s_\beta(\Delta)$, we have $A \in (bv_p(\alpha), s_\beta(\Delta))$ if and only if $D_{1/\beta} \Delta A \in (bv_p(\alpha), l_\infty)$. Since $bv_p(\alpha) = \Sigma l_p(\alpha)$, this means

$$(D_{\frac{1}{\beta}} \Delta A)(\Sigma X) \in l_\infty \text{ for all } X \in l_p(\alpha).$$

By Lemma 8(ii), condition (16) implies $(D_{1/\beta} \Delta A)(\Sigma X) = (D_{1/\beta} \Delta A \Sigma)X$ for all $X \in l_p(\alpha)$, and $A \in (bv_p(\alpha), s_\beta(\Delta))$ if and only if $D_{1/\beta} \Delta A \Sigma \in (l_p(\alpha), l_\infty)$, which in turn is (17).

(ii)(b) If $\beta \in \Gamma$ then by Lemma 5(ii) $bv_\infty(\beta) = s_\beta(\Delta) = s_\beta$. As above under condition (18) $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if $D_{1/\beta} A \Sigma \in (l_p(\alpha), l_\infty)$. This gives the conclusion.

(iii) Here $bv_\infty(\alpha) = l_\infty(\alpha)(\Delta) = s_\alpha(\Delta)$ and $bv_\infty(\beta) = s_\beta(\Delta)$. As above it can easily be seen that $A \in (s_\alpha(\Delta), s_\beta(\Delta))$ if and only if $\Delta A \Sigma \in S_{\alpha,\beta}$, under condition (19).

We also have the following results when $\alpha \in \Gamma$.

Proposition 14.

(i) *If $\alpha \in \Gamma$, then $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if*

$$\begin{cases} \sup_{n,m} \left(\dfrac{1}{\beta_n} |a_{nm} - a_{n-1,m}| \alpha_m \right) < \infty & for\ p = 1, \\ \sup_n \left[\dfrac{1}{\beta_n^q} \displaystyle\sum_{m=1}^{\infty} (|a_{nm} - a_{n-1,m}| \alpha_m)^q \right] < \infty & for\ 1 < p < \infty. \end{cases}$$

(ii) *If $\alpha, \beta \in \Gamma$, then $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if*

$$\begin{cases} \sup_{n,m} \left(\dfrac{1}{\beta_n} |a_{nm}| \alpha_m \right) < \infty & for\ p = 1, \\ \sup_n \left[\dfrac{1}{\beta_n^q} \displaystyle\sum_{m=1}^{\infty} (|a_{nm}| \alpha_m)^q \right] < \infty & for\ 1 < p < \infty. \end{cases}$$

Proof. Since $\alpha \in \Gamma$ we have $bv_p(\alpha) = l_p(\alpha)$ and $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if $\Delta A \in (l_p(\alpha), s_\beta)$. Now the condition $\Delta A \in (l_p(\alpha), s_\beta)$ means that $D_{1/\beta}\Delta A D_\alpha \in (l_p, l_\infty)$ and we conclude by Lemma 7.

(ii) The condition $\alpha, \beta \in \Gamma$ implies $bv_p(\alpha) = l_p(\alpha)$ and $bv_\infty(\beta) = s_\beta(\Delta) = s_\beta$. Thus $A \in (bv_p(\alpha), bv_\infty(\beta))$ if and only if $D_{1/\beta} A D_\alpha \in (l_p, l_\infty)$, and we conclude by Lemma 7.

Study now the set $(bv_\infty(\alpha), bv_u(\beta))$. We obtain

Proposition 15. *Let* $1 \leq u < \infty$.

(i) *Under the condition*

$$\sum_{m=1}^{\infty} \alpha_m \sum_{j=m}^{\infty} |a_{nj} - a_{n-1,j}| < \infty \qquad \textit{for all } n, \tag{21}$$

$A \in (bv_\infty(\alpha), bv_u(\beta))$ *if and only if*

$$\sup_{N \subset \mathbb{N}, N \text{ finite}} \left(\sum_{n=1}^{\infty} \left| \sum_{k \in N} \frac{1}{\beta_n} \sum_{m=k}^{\infty} (a_{nm} - a_{n-1,m})\alpha_m \right|^u \right) < \infty.$$

(ii) *Let* $\alpha \in \Gamma$. *Then* $A \in (bv_\infty(\alpha), bv_u(\beta))$ *if and only if*

$$\sup_{N \subset \mathbb{N}, N \text{ finite}} \left(\sum_{n=1}^{\infty} \left| \frac{1}{\beta_n} \sum_{m \in N} (a_{nm} - a_{n-1,m})\alpha_m \right|^u \right) < \infty; \tag{22}$$

(iii) *If* $\beta \in \Gamma$, *under the condition*

$$\sum_{m=1}^{\infty} \alpha_m \sum_{k=m}^{\infty} |a_{nk}| < \infty \qquad \textit{for all } n, \tag{23}$$

$A \in (bv_\infty(\alpha), bv_u(\beta))$ *if and only if*

$$\sup_{N \subset \mathbb{N}, N \text{ finite}} \left(\sum_{n=1}^{\infty} \left| \frac{1}{\beta_n} \sum_{k \in N} \sum_{m=k}^{\infty} a_{nm}\alpha_m \right|^u \right) < \infty.$$

(iv) *Let* $\alpha, \beta \in \Gamma$. *Then* $A \in (bv_\infty(\alpha), bv_u(\beta))$ *if and only if*

$$\sup_{N \subset \mathbb{N}, N \text{ finite}} \left(\sum_{n=1}^{\infty} \left| \frac{1}{\beta_n} \sum_{m \in N} a_{nm}\alpha_m \right|^u \right) < \infty.$$

Proof.

(i) $A \in (bv_\infty(\alpha), bv_u(\beta))$ if and only if $\Delta A \in (s_\alpha(\Delta), l_u(\beta))$. For all $X \in s_\alpha$

$$\Delta A(\Sigma X) \in l_u(\beta).$$

Now since (21) holds $\Delta A(\Sigma X) = (\Delta A \Sigma)X$ for all $X \in s_\alpha$. Then $A \in (bv_\infty(\alpha), bv_u(\beta))$ if and only if $D_{1/\beta}\Delta A \Sigma D_\alpha \in (l_\infty, l_u)$, and we conclude applying Lemma 7.

(ii) Since $\alpha \in \Gamma$ we get $bv_\infty(\alpha) = s_\alpha(\Delta) = s_\alpha$. So $A \in (bv_\infty(\alpha), bv_u(\beta))$ if and only if

$$\Delta A \in (s_\alpha, l_u(\beta)),$$

that is $D_{1/\beta}\Delta A D_\alpha \in (l_\infty, l_u)$ and we conclude as above.

(iii) Here $bv_u(\beta) = l_u(\beta)$ and $A \in (s_\alpha(\Delta), l_u(\beta))$ if and only if $A(\Sigma X) \in l_u(\beta)$ for all $X \in s_\alpha$. Since (23) holds we have $A(\Sigma X) = (A\Sigma)X$ for all $X \in s_\alpha$ and $A \in (s_\alpha(\Delta), l_u(\beta))$ if and only if $A\Sigma \in (s_\alpha, l_u(\beta))$, that is $D_{1/\beta}A\Sigma D_\alpha \in (l_\infty, l_u)$. we conclude applying Lemma 7(ii).

(iv) Now $(bv_\infty(\alpha), bv_u(\beta)) = (s_\alpha, l_u(\beta))$ and $A \in (s_\alpha, l_u(\beta))$ if and only if $D_{1/\beta}AD_\alpha \in (l_\infty, l_u)$ and we conclude by Lemma 7.

Remark 8. Note that in Proposition 15(ii), for $h \geq 1$ integer and $\alpha \in \Gamma$, we have $A \in (bv_\infty^h(\alpha), bv_u(\beta)$ if and only if (22) holds.

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