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NEIGHBORHOOD CONNECTED PERFECT DOMINATION IN GRAPHS

P. SELVARAJU, M. P. KULANDAIVEL AND C. SIVAGNANAM

Abstract. Let $G = (V, E)$ be a connected graph. A set S of vertices in G is a perfect dominating set if every vertex v in $V-S$ is adjacent to exactly one vertex in S. A perfect dominating set S is said to be a neighborhood connected perfect dominating set (ncpd-set) if the induced subgraph $< N(S) >$ is connected. The minimum cardinality of a ncpd-set of G is called the neighborhood connected perfect domination number of G and is denoted by $\gamma_{ncp}(G)$. In this paper we initiate a study of this parameter.

1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2] and Haynes et al. [3, 4].

For any $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$, then the private neighbor set of u with respect to S is defined by $\text{pn}[u, S] = \{v : N[v] \cap S = \{u\}\}$. The chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colours required to colour all the vertices such that no two adjacent vertices receive the same colour.

A subset S of V is called a dominating set if every vertex u in $V-S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al., P.M. Weichsel [see 3] introduced the concept of perfect domination in graphs. A dominating set S of G is called a perfect dominating set if every vertex v in $V-S$ is adjacent to exactly one vertex in S. The minimum cardinality of a perfect dominating set is called perfect domination number of G and is denoted by $\gamma_p(G)$. S. Arumugam and C. Sivagnanam [1]

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introduced the concept of neighborhood connected domination in graphs. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $< N(S) >$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. In this paper we introduce the concept of neighborhood connected perfect domination and initiate a study of the corresponding parameter. We need the following theorems.

Theorem 1.1 ([1]). *For a path P_n, $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 1.2 ([1]). *For the cycle C_n on n vertices*

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil & \text{if } n \not\equiv 3 \pmod 4 \\ \left\lfloor \dfrac{n}{2} \right\rfloor & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

2. Main results

Definition 2.1. A perfect dominating set S of a graph G is called the neighborhood connected perfect dominating set (ncpd-set) if the induced subgraph $< N(S) >$ is connected. The minimum cardinality of a ncpd-set of G is called the neighborhood connected perfect domination number of G and is denoted by $\gamma_{ncp}(G)$.

Remark 2.2. (i) Clearly $\gamma_{ncp}(G) \geq \gamma_{nc}(G) \geq \gamma(G)$.

(ii) For any connected graph G, $\gamma_{ncp}(G) = 1$ if and only if there exists a non cut vertex v such that $\deg v = n-1$. Thus $\gamma_{ncp}(G) = 1$ if and only if $G = H + K_1$ for some connected graph H.

(iii) For a tree T with $n \geq 3$, $\gamma_{ncp}(T) \geq 2$.

Theorem 2.3. *For any path P_n, $\gamma_{ncp}(P_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $P_n = (v_1, v_2, \cdots, v_n)$. If $n \not\equiv 1 \pmod 4$ then $S = \{v_i : i = 2k, 2k+1 \text{ and } k \text{ is odd}\}$ is a ncpd-set of P_n and if $n \equiv 1 \pmod 4$ then $S_1 = S \cup \{v_{n-1}\}$ is a ncpd-set of P_n. Hence $\gamma_{ncp}(P_n) \leq \lceil \frac{n}{2} \rceil$. Since $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$ and $\gamma_{ncp}(G) \geq \gamma_{nc}(G)$, we have $\lceil \frac{n}{2} \rceil \leq \gamma_{ncp}(P_n)$. Thus $\gamma_{ncp}(P_n) = \lceil \frac{n}{2} \rceil$. □

Corollary 2.4. *For any non trivial path P_n,* (i) *$\gamma_{ncp}(P_n) = \gamma(P_n)$ if and only if $n = 2$ or 4.* (ii) *$\gamma_{ncp}(P_n) = \gamma_p(P_n)$ if and only if $n = 2$ or 4.*

Proof. Since $\gamma(P_n) = \gamma_p(P_n) = \lceil \frac{n}{3} \rceil$ the corollary follows. □

Theorem 2.5.

$$\gamma_{ncp}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \textit{if} \quad n \equiv 0, 1 \pmod 4 \\ \frac{n}{2} + 1 & \textit{if} \quad n \equiv 2 \pmod 4 \\ \lfloor \frac{n}{2} \rfloor & \textit{if} \quad n \equiv 3 \pmod 4 \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \cdots, v_n, v_1)$ and $n = 4k + r$, where $0 \leq r \leq 3$. Let $S = \{v_i : i = 2j, 2j + 1, j$ is odd and $1 \leq j \leq 2k - 1\}$

$$\text{Let} \quad S_1 = \begin{cases} S & \text{if} \quad n \equiv 0 \pmod 4 \\ S \cup \{v_{n-1}\} & \text{if} \quad n \equiv 1, 3 \pmod 4 \\ S \cup \{v_{n-2}, v_{n-1}\} & \text{if} \quad n \equiv 2 \pmod 4 \end{cases}$$

Clearly S_1 is a ncpd-set of C_n and hence

$$\gamma_{ncp}(C_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil & \text{if} \quad n \equiv 0, 1 \pmod 4 \\ \frac{n}{2} + 1 & \text{if} \quad n \equiv 2 \pmod 4 \\ \lfloor \frac{n}{2} \rfloor & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

Since $\gamma_{ncp}(C_n) \geq \gamma_{nc}(C_n)$ and

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \not\equiv 3 \pmod 4 \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

it follows that values given for $\gamma_{ncp}(C_n)$ are correct unless $n \equiv 2$ (($\bmod 4$). If $n \equiv 2 \pmod 4$, then for any γ_{nc}-set S of C_n, there exists a vertex $v \in V - S$ adjacent to two vertices in S and hence $\gamma_{ncp}(C_n) \geq \frac{n}{2} + 1$.

Hence the result follows. □

Corollary 2.6. (i) $\gamma_{ncp}(C_n) = \gamma(C_n)$ *if and only if* $n = 3, 4, or 7$. (ii) $\gamma_{ncp}(C_n) = \gamma_p(C_n)$ *if and only if* $n = 3, 4, 5, 7 or 8$. (iii) $\gamma_{ncp}(C_n) = \gamma_{nc}(C_n)$ *if* $n \not\equiv 2 \pmod 4$.

Proof. Since $\gamma(C_n) = \lceil \frac{n}{3} \rceil$,

$$\gamma_p(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 2 \pmod 3 \\ \left\lceil \frac{n}{3} \right\rceil & \text{otherwise} \end{cases}$$

the result follows. □

Theorem 2.7. *Let S be a minimal ncpd-set of a graph G. Then for every $u \in S$, one of the following holds* (i) $pn[u, S] \neq \phi$. (ii) $|N(u) \cap (S - \{u\})| \geq 2$. (iii) $< N(S - \{u\}) >$ *is disconnected.*

Proof. Let S be a minimal ncpd-set of G. Let $u \in S$ and let $S_1 = S - \{u\}$. Then any one of the following is true.

(a) S_1 is not a dominating set. (b) $< N(S_1) >$ is disconnected. (c) there exists a vertex $v \in V - S_1$ such that $|N(v) \cap S_1| \geq 2$.

If $< N(S_1) >$ is disconnected then (iii) is true. If S_1 is not a dominating set of G, then pn$[u, S] \neq \phi$. Suppose a vertex $v \in V - S_1$, such that $|N(v) \cap S_1| \geq 2$. If $v \neq u$ then there exist two vertices $x, y \in S_1$ such that x, y are adjacent to v and hence S is not a ncpd-set. Thus $v = u$ which gives (ii) of the theorem. □

Theorem 2.8. *Let G be a graph with $\Delta = n-1$ and let $v \in V(G)$ with* $\deg v = \Delta$. *Then $\gamma_{ncp}(G) \leq 1 + |V(H)|$ where H is a component of $G - v$ with $|V(H)|$ is minimum.*

Proof. Let $v \in V(G)$ with $\deg v = n-1$. If $G - v$ is connected then $\{v\}$ is a ncpd-set of G and hence $\gamma_{ncp}(G) = 1$. Suppose $G - v$ is disconnected, then $S = \{v\}$ is not a ncpd-set of G. Let H be a component of $G - v$ with minimum vertices. Hence $S \cup V(H)$ is a ncpd-set of G. Thus $\gamma_{ncp}(G) \leq 1 + |V(H)|$. □

Remark 2.9. The bound given in Theorem 2.8 is sharp. The graph $G = K_{1,n-1}$, $\gamma_{ncp}(G) = 2 = 1 + |V(H)|$.

Corollary 2.10. *Let G be a graph with $\Delta = n-1$. Then $\gamma_{ncp}(G) = 2$ if and only if there exists a support vertex v such that* $\deg v = n-1$.

Theorem 2.11. *Let G be any graph and H be a connected spanning subgraph of G with $\gamma_{ncp}(G) > \gamma_{ncp}(H)$. Then $\gamma_{ncp}(G) > \gamma_{nc}(G)$.*

Proof. Suppose $\gamma_{ncp}(G) = \gamma_{nc}(G)$. Since $\gamma_{nc}(G) \leq \gamma_{nc}(H)$ we have $\gamma_{ncp}(G) \leq \gamma_{ncp}(H)$ which is a contradiction. This proves the result. □

Theorem 2.12. *For any graph G, $\gamma_{ncp}(G) \leq n$. Further, if G is a $(n-2)$-regular graph, $n \geq 6$, then $\gamma_{ncp}(G) = n$.*

Proof. First part is obvious. Suppose G is $(n-2)$-regular and let S be any γ_{ncp}-set of G. Clearly S contains at least two vertices. Suppose $\gamma_{ncp}(G) < n$.

Case (i). $\gamma_{ncp}(G) = 2$

Then there exists a vertex $x \in V - S$ which is adjacent to vertices of S which is a contradiction.

Case (ii). $3 \leq \gamma_{ncp}(G) \leq n-1$

Then $w \in V - S$ is adjacent to at least two vertices of S which is a contradiction. Hence $\gamma_{ncp}(G) = n$. □

Problem 2.13. Characterize the class of graphs for which $\gamma_{ncp}(G) = n$.

Theorem 2.14. *Let G be a graph with k pendant vertices. Then* $\gamma_{ncp}(G) \leq n-k+1$ *and equality holds if and only if G is a star.*

Proof. Let X be the set of all pendant vertices of a graph G and let $|X| = k$. Let $u \in X$. Then $(V - X) \cup \{u\}$ is a ncpd-set of G. Hence $\gamma_{ncp}(G) \leq n - k + 1$. Let G be a graph with $\gamma_{ncp}(G) = n - k + 1$ and let X be the set of all pendant vertices of G with $|X| = k$. If $|V - X| > 1$ then $V - X$ is a ncpd-set of G with $|V - X| = n - k$ which is a contradiction. Hence $|V - X| = 1$. Thus G is a star. □

Problem 2.15. Characterize the class of graphs for which $\gamma_{ncp}(G) = n - k$ where k is the number of pendant vertices in G.

In the next two theorems we find an upper bound for sum of the neighborhood connected perfect domination number and chromatic number and characterize the corresponding extremal graphs.

Theorem 2.16. *For any nontrivial graph G,* $\gamma_{ncp}(G) + \chi(G) \leq 2n - 1$ *and equality holds if and only if G is isomorphic to* K_2.

Proof. Suppose $\gamma_{ncp}(G) + \chi(G) = 2n$ then $\gamma_{ncp}(G) = n$ and $\chi(G) = n$. Then G is a complete graph with $\gamma_{ncp}(G) = n$ which gives G is trivial and hence $\gamma_{ncp}(G) + \chi(G) \leq 2n - 1$.

Let G be a graph with $\gamma_{ncp}(G) + \chi(G) = 2n - 1$. Then either $(i)\, \gamma_{ncp}(G) = n - 1$, $\chi(G) = n$ or $(ii)\, \gamma_{ncp}(G) = n$, $\chi(G) = n - 1$. Suppose (i) holds. Then G is a complete graph with $\gamma_{ncp}(G) = n - 1$ which gives $n = 2$. Hence G is isomorphic to K_2. Suppose (ii) holds. Then G is a isomorphic to $K_n - X$ where X is a non empty subset of set of edges incident with a vertex v of K_n with $|X| \leq n - 2$ which implies $\gamma_{ncp}(G) = 1$ or 2. Then $n = 2$ and hence G is disconnected which is a contradiction. The converse is obvious. □

Theorem 2.17. *Let G be a graph. Then* $\gamma_{ncp}(G) + \chi(G) = 2n - 2$ *if and only if G is isomorphic to* K_3 *or* P_3 *or the graph obtained from* $K \cup H$ *where* $K = K_{n-2}$ *and H is either* K_2 *or* $\overline{K_2}$ *with* $V(H) = \{u, v\}$ *by adding* n_1 *edges between u and K and adding* n_2 *edges between v and K,* $2 \leq n_i \leq n - 5$, $i = 1$ *or* 2, *such that* $[N(u) \cap N(v)] - \{u, v\} = \phi$ *and* $n_1 + n_2 < n - 2$.

Proof. Let $\gamma_{ncp}(G) + \chi(G) = 2n - 2$. Then one of the following is true $(i)\, \gamma_{ncp}(G) = n - 2$, $\chi(G) = n$ $(ii)\, \gamma_{ncp}(G) = n - 1$, $\chi(G) = n - 1$ $(iii)\, \gamma_{ncp}(G) = n$, $\chi(G) = n - 2$.

Suppose (i) holds. Then G is a complete graph with $\gamma_{ncp}(G) = n - 2$ this implies $n = 3$. Hence G is isomorphic to K_3. Suppose (ii) holds. Then G is isomorphic to $K_n - X$, where X is a non empty subset of set of edges incident with a vertex of K_n with $|X| \leq n - 2$ which implies $\gamma_{ncp}(G) = 1$ or 2. Then $n = 2$ or 3 and hence G is isomorphic to P_3. Suppose (iii) holds.

Because $\chi(G) = n-2$, either G has a complete subgraph of order $n-2$ or $n > 4$ and G is the join of K_{n-5} with C_5. (In case $n = 5$, by the join of K_{n-5} and C_5 we mean C_5.) If G is the join of K_{n-5} with C_5 then $\gamma_{ncp}(G)+\chi(G) = 6$, if $n = 5$, or $n-1$, if $n > 5$. In either case, $\gamma_{ncp}(G)+\chi(G) \neq 2n-2$. Thus G has a complete subgraph G_1 of order $n-2$. Let $Y = V(G) - V(G_1) = \{u, v\}$. Then $\langle Y\rangle = K_2$ or $\overline{K_2}$.

Case (i). $\langle Y\rangle = \overline{K_2}$

Since G is a connected graph each u and v are adjacent to at least one vertex of G_1. If either u or v is a pendant vertex, then $\gamma_{ncp}(G) < n$. Hence each u and v are adjacent to at least two vertices in G_1. If u and v have a common neighbor w in G_1, then $\gamma_{ncp}(G) = 1$ which gives a contradiction. Hence $N(u)\cap N(v) = \phi$. If $N(u)\cup N(v) = V(G_1)$ then $\gamma_{ncp}(G) = 2$ which is a contradiction. Then the graph is isomorphic to the graph given in theorem.

Case (ii). $\langle Y\rangle = K_2$.

Since G is connected and $\gamma_{ncp}(G) = n$ we have each u and v are adjacent to at least one vertex of G_1. If u and v have a common neighbor w in G_1, then $\gamma_{ncp}(G) = 1$ or 3 which gives a contradiction. Hence $N(u)\cap N(v) = \phi$. Suppose $N(u)\cap V(G_1) = \{x\}$ then $\{u, x\}$ is a γ_{ncp}-set G which is a contradiction. Hence each u and v are adjacent to more than one vertex in G_1.

If $[N(u)\cap N(v)]-\{u, v\} = V(G_1)$ then $\gamma_{ncp}(G) = 2$ which is a contradiction. Then the graph is isomorphic to the graph given in theorem. The converse is obvious. □

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Department of Mathematics, VELTECH (Owned by RS Trust), No. 60, Veltech-Avadi Road, Chennai-600062, Tamilnadu, India.

E-mail: pselvar@yahoo.com

Department of Mathematics, St. Joseph's College of Engineering, Chennai-600119, India.

E-mail: gracempk@yahoo.co.in

School of Sciences, Birla Institute of Technology, Kingdom of Bahrain.

E-mail: choshi71@gmail.com