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# UNIQUENESS AND VALUE SHARING OF MEROMORPHIC FUNCTIONS WITH REGARD TO MULTIPLICITY

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**Abstract**. In this paper, we study the uniqueness theorems of meromorphic functions, concerning differential polynomials and obtain theorems, from which we obtain as a very special case the results of Lin and Yi [4], Xiao Yu Zhang, Jun-Fan Chen, Wei-Chuan lin [8], and Renukadevi S. Dyavanal [9]. We also obtain several new interesting results.

## 1. Introduction And Main Results

In this paper the term meromorphic will always mean meromorphic in the complex plane. Let $f$ and $g$ be non-constant meromorphic functions and $a$ be a complex number. We say $f$ and $g$ share the value $a$ CM, if $f-a$ and $g-a$ have the same zeros with the same multiplicities. It is assumed that reader is familiar with notations of Nevanlinna theory of meromorphic functions, for instance, $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, etc (see [1, 3]). We denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$. as $r\to+\infty$, possibly outside a set of finite measure.

In 2004, Lin and Yi [4] proved the following theorems.

**Theorem A.** *Let $f$ and $g$ be two non-constant meromorphic functions, $n\geq 12$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g=(n+2)(1-h^{n+1})/(n+1)(1-h^{n+2}), f=(n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$, where $h$ is a non-constant meromorphic function.*

**Theorem B.** *Let $f$ and $g$ be non-constant meromorphic functions, $n\geq 13$ be a positive integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f(z)\equiv g(z)$.*

In 2008, Xiao-Yu Zhang, Jun-Fan Chen, Wei-Chuan Lin [8] extended Theorems A and B and proved the following theorem.

**Theorem C.** *Let $f$ and $g$ be two nonconstant meromrophic functions, let $n$ and $m$ be two positive integers with $n>max\{m+10,3m+3\}$ and let $P(z)=a_mz^m+a_{m-1}z^{m-1}+\cdots+a_0$, where $a_0\neq$*

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*0, $a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share 1 CM, then either $f \equiv tg$, for a constant $t$ such that $t^d = 1$, where $d = \{n+m+1, \ldots, n+m+1-i, \ldots, n+1\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$ or $f$ and $g$ satisfy algebraic equation $R(f, g) = 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^{n+1}\left(\frac{a_m \omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right) - \omega_2^{n+1}\left(\frac{a_m \omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right).$$

In 2004, Wei-Chuan Lin and Hong Xun Yi [7], extended Theorems A and B by replacing the value 1 with the function $z$ and obtained the following results.

**Theorem D.** *Let $f$ and $g$ be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, then either $f(z) \equiv g(z)$ or $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2}), f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$, where $h$ is a non-constant meromorphic function.*

**Theorem E.** *Let $f$ and $g$ be transcendental meromorphic functions, $n \geq 13$ is an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $z$ CM, then $f(z) \equiv g(z)$.*

In 2009, Hong Yan Xu and Ting Bin Cao [6], obtained the following result.

**Theorem F.** *Let $f$ and $g$ be two transcendental meromorphic functions and let $n$ and $m$ be two positive integers with $n > m+10$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$, where $a_0 \neq 0$, $a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share $z$ CM, then conclusion of Theorem C still holds.*

In 2011, Renukadevi S. Dyavanal [9] proved the following results.

**Theorem G.** *Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2}), f = (n+2)h(1-h^{n+1})/(n+1)(1-h^{n+2})$, where $h$ is a non-constant meromorphic function.*

**Theorem H.** *Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n$ be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f \equiv g$.*

In this paper, using notion of multiplicity, we prove the following two theorems. As a consequence of these theorems, we improve the above mentioned theorems and in addition, we also obtain some new interesting results.

**Theorem 1.1.** *Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n, m$ be positive integers with $(n-m-1)s \geq max\{10, 2m+3\}$ and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$, where $a_0 \neq 0, a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f) f'$ and $g^n P(g) g'$ share 1 CM, then one of the following two cases holds:*

(1) *$f = tg$ for a constant $t$ such that $t^d = 1$, where $d = \{n+m+1, \ldots, n+m+1-i, \ldots, n+1\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*

(2) *$f$ and $g$ satisfy algebraic equation $R(f, g) \equiv 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^{n+1}\left(\frac{a_m \omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right) - \omega_2^{n+1}\left(\frac{a_m \omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right).$$

**Remark 1.1.** We set $P(z) = (z-1)^m$. Then with $a_m = 1, a_0 = -1$ and under the condition (2) of Theorem 1.1, Theorem 1.1 reduces to

(i) Theorem G, if $m = 1$,

(ii) Theorem A, if $m = 1$ and $s = 1$,

(iii) Theorem H, if $m = 2$,

(iv) Theorem B, if $m = 2$ and $s = 1$,

(v) Theorem C, if $s = 1$.

**Theorem 1.2.** *Let $f$ and $g$ be two transcendental meromorphic functions, whose zeros and poles are of multiplicities atleast $s$, where $s$ is a positive integer. Let $n, m$ be positive integers with $(n-m-1)s \geq max\{10, 2m+3\}$. Let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$, where $a_0 \neq 0, a_1, a_2, \ldots, a_{m-1}, a_m \neq 0$ are complex constants. Let $f^n P(f) f'$ and $g^n P(g) g'$ share $z$ CM, then conclusion of Theorem 1.1 still holds.*

**Remark 1.2.** We set $P(z) = (z-1)^m$. Then with $a_m = 1, a_0 = -1$ and under the condition (2) of Theorem 1.2, Theorem 1.2 reduces to

(i) Theorem D, if $m = 1$, $s = 1$,

(ii) Theorem E, if $m = 2$, $s = 1$,

(iii) Theorem F, if $s = 1$.

Some interesting new results in this vein are indicated in Section 4.

## 2. Some Lemmas

**Lemma 1** ([2]). *Let $f$ be non-constant meromorphic function and let $a_n(\neq 0), a_{n-1}, \ldots, a_0$ be small functions with respect to $f$. Then*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0\right) = nT(r,f) + S(r,f)$$

**Lemma 2** ([5]). *Let $f$ and $g$ be two non-constant meromorphic functions. If $f$ and $g$ share 1 CM, one of the following three cases holds:*

(i) $T(r,f) \leq N_2(r,f) + N_2(r,g) + N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g)$ *the same inequality holding for $T(r,g)$;*

(ii) $f \equiv g$;

(iii) $fg \equiv 1$.

**Lemma 3** ([10]). *Let $Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)(\omega^{n-1} - 1)^2$, then*

$$Q(\omega) = (\omega - 1)^4(\omega - \beta_1)(\omega - \beta_2)\cdots(\omega - \beta_{2n-6}),$$

*where $\beta_j \in \mathbb{C} \setminus \{0,1\}$, $(j = 1,2,\ldots,2n-6)$, which are distinct respectively.*

**Lemma 4** ([2]). *Suppose $f(z)$ is a nonconstant meromorphic function in the complex plane and $k$ is a positive integer.Then*

$$N(r,\frac{1}{f^{(k)}}) \leq N(r,\frac{1}{f}) + k\overline{N}\left(r,f\right) + S(r,f).$$

## 3. Proof of Theorems 1.1 and 1.2

### 3.1. Proof of Theorem 1.1.

Let

$$F = f^n P(f) f', \qquad G = g^n P(g) g' \tag{1}$$

and

$$\begin{aligned} F^* &= \frac{a_m f^{m+n+1}}{m+n+1} + \frac{a_{m-1} f^{m+n}}{m+n} + \cdots + \frac{a_0 f^{n+1}}{n+1} \\ G^* &= \frac{a_m g^{m+n+1}}{m+n+1} + \frac{a_{m-1} g^{m+n}}{m+n} + \cdots + \frac{a_0 g^{n+1}}{n+1} \end{aligned} \tag{2}$$

By hypothesis $F$ and $G$ share 1 CM. By Lemma 1, we have

$$\begin{aligned} T(r,F^*) &= (n+m+1)T(r,f) + S(r,f) \\ T(r,G^*) &= (n+m+1)T(r,g) + S(r,f) \end{aligned} \tag{3}$$

Since $(F^*)' = F$, we deduce,

$$m\left(r,\frac{1}{F^*}\right) \le m\left(r,\frac{1}{F}\right) + S(r,f),$$

and by the first fundamental Theorem,

$$\begin{aligned} T(r,F^*) \le\ & T(r,F) + N\left(r,\frac{1}{F^*}\right) - N\left(r,\frac{1}{F}\right) + S(r,f) \\ \le\ & T(r,F) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-b_1}\right) + \cdots + N\left(r,\frac{1}{f-b_m}\right) \\ & -N\left(r,\frac{1}{f-c_1}\right) - \cdots - N\left(r,\frac{1}{f-c_m}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f) \end{aligned} \tag{4}$$

where $b_1, b_2, \ldots, b_m$ are roots of algebraic equation $\frac{a_m z^m}{m+n+1} + \frac{a_{m-1} z^{m-1}}{m+n} + \cdots + \frac{a_0}{n+1} = 0$ and $c_1, c_2, \ldots,$ $c_m$ are roots of the algebraic equation $a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0 = 0$.

By Lemma 2, one of the following three cases holds:

**Case 1:**

$$T(r,F) \le N_2(r,F) + N_2\left(r,\frac{1}{F}\right) + N_2(r,G) + N_2\left(r,\frac{1}{G}\right) + S(r,f) + S(r,g), \tag{5}$$

the same inequality holding for $T(r,G)$.

On the other hand, we have

$$N_2(r,F) + N_2\left(r,\frac{1}{F}\right) \le 2\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-c_1}\right) + \cdots + N\left(r,\frac{1}{f-c_m}\right) + N\left(r,\frac{1}{f'}\right) \tag{6}$$

$$N_2(r,G) + N_2\left(r,\frac{1}{G}\right) \le 2\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g-c_1}\right) + \cdots + N\left(r,\frac{1}{g-c_m}\right) + N\left(r,\frac{1}{g'}\right) \tag{7}$$

From (3)−(7), we obtain

$$\begin{aligned} T(r,F^*) \le\ & 2\overline{N}(r,f) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}(r,g) + 2\overline{N}\left(r,\frac{1}{g}\right) \\ & +N\left(r,\frac{1}{g-c_1}\right) + \cdots + N\left(r,\frac{1}{g-c_m}\right) \\ & +N\left(r,\frac{1}{g'}\right) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-b_1}\right) + \cdots + N\left(r,\frac{1}{f-b_m}\right) + S(r,f) + S(r,g). \end{aligned}$$

By lemma 4, $N\left(r,\frac{1}{g'}\right) \le N\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + S(r,g)$ and by our assumption, zeros and poles of $f$ and $g$ are of multiplicities atleast $s$, we have

$$\overline{N}(r,g) \le \frac{1}{s}N(r,g) \le \frac{1}{s}T(r,g) \tag{8}$$

and

$$\overline{N}\left(r,\frac{1}{g}\right) \le \frac{1}{s}N\left(r,\frac{1}{g}\right) \le \frac{1}{s}T(r,g), \tag{9}$$

we deduce above inequality as,

$$T(r,F^*) \leq \left(\frac{4}{s}+m+1\right)T(r,f)+\left(\frac{5}{s}+m+1\right)T(r,g)+S(r,f)+S(r,g)$$
$$(n+m+1)T(r,f) \leq \left(\frac{4}{s}+m+1\right)T(r,f)+\left(\frac{5}{s}+m+1\right)T(r,g)+S(r,f)+S(r,g)$$
$$\left(n-\frac{4}{s}\right)T(r,f) \leq \left(\frac{5}{s}+m+1\right)T(r,g)+S(r,f)+S(r,g) \tag{10}$$

Similarly

$$\left(n-\frac{4}{s}\right)T(r,g) \leq \left(\frac{5}{s}+m+1\right)T(r,f)+S(r,f)+S(r,g). \tag{11}$$

From (10) and (11), we deduce that $(n-m-1)s \leq 9$, which contradicts $(n-m-1)s \geq 10$.

**Case 2:** Suppose that $FG \equiv 1$, that is

$$f^n P(f) f' \; g^n P(g) g' \equiv 1. \tag{12}$$

Now we rewrite $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ as

$$P(z) = a_m (z-d_1)^{l_1} (z-d_2)^{l_2} \cdots (z-d_i)^{l_i} \cdots (z-d_k)^{l_k},$$

where $l_1 + l_2 + \cdots + l_k = m$, $1 \leq k \leq m$; $d_i \neq d_j$, $i \neq j$, $i, j \leq k$; $d_1, d_2, \cdots, d_k$ are non-zero constants and $l_1, l_2, \ldots, l_k$ are positive integers.

Let $z_0$ be a zero of $f$ of order $p$. Then from (12) we know that $z_0$ is a pole of $g$. Suppose $z_0$ is a pole of $g$ of order $q$. Again by (12), we obtain

$$np + p - 1 = nq + mq + q + 1,$$

that is

$$(n+1)(p-q) = mq + 2.$$

which implies $p \geq q+1$ and $mq+2 \geq n+1$. Hence

$$p \geq \frac{n+m-1}{m} \tag{13}$$

Let $z_1$ be a zero of $P(f)$ of order $p_1$ and a zero of $f - d_i$ of order $q_i$ for $i = 1, 2, \ldots, k$. Then $p_1 = l_i q_i$ for $i = 1, 2, \ldots, k$. Suppose that $z_1$ is a pole of $g$ of order $t_1$. Again by (12), we obtain $q_i l_i + q_i - 1 = nt_1 + mt_1 + t_1 + 1$. That is,

$$q_i = \frac{nt_1 + mt_1 + t_1 + 2}{(l_i + 1)}$$
$$q_i \geq \frac{ns + ms + s + 2}{l_i + 1}, \qquad for \quad i = 1, 2, \ldots, k \tag{14}$$

Let $z_2$ be a zero of $f'$ of order $p_2$, that is not a zero of $fP(f)$. Similarly, we get $p_2 = nt_2 + mt_2 + t_2 + 1$. That is,

$$p_2 \geq ns + ms + s + 1 \tag{15}$$

In the same manner as above, we have similar results for the zeros of $g^nP(g)g'$.

From (12), we can write,

$$\begin{aligned}
\overline{N}\left(r, f^nP(f)f'\right) &= \overline{N}\left(r, \frac{1}{g^nP(g)g'}\right) \\
\overline{N}\left(r, f\right) &= \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}\left(r, \frac{1}{g'}\right) \\
\overline{N}(r, f) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-d_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{g-d_k}\right) + \overline{N_0}\left(r, \frac{1}{g'}\right)
\end{aligned}$$

From (13)−(15), we obtain

$$\overline{N}(r, f) \leq \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, g) + N_0\left(r, \frac{1}{g'}\right) + S(r, g) \tag{16}$$

where $N_0\left(r, \frac{1}{g'}\right)$ denotes the counting function corresponding to the zeros of $g'$ that are not the zeros of $gP(g)$ and $N_0\left(r, \frac{1}{f'}\right)$ denotes the analogous quantity,$\left(\overline{N_0}\left(r, \frac{1}{g'}\right)\right)$ denotes the reduced counting function.

Similarly , as above we can obtain,

$$\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-d_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{f-d_k}\right) \leq \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, f).$$

By the second fundamental theorem and from (16), we have

$$\begin{aligned}
kT(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-d_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{f-d_k}\right) + \overline{N}(r, f) - N_0\left(r, \frac{1}{f'}\right) + S(r, f) \\
&\leq \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, f) + \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, g) \\
&\quad + N_0\left(r, \frac{1}{g'}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r, f).
\end{aligned} \tag{17}$$

Similarly , we have

$$\begin{aligned}
kT(r, g) &\leq \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, g) + \left(\frac{m}{n+m-1} + \frac{m+k}{ns+ms+s+2}\right) T(r, f) \\
&\quad + N_0\left(r, \frac{1}{f'}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r, g).
\end{aligned} \tag{18}$$

From (17) and (18), we have

$$k(T(r, f) + T(r, g)) \leq \left(\frac{2m}{n+m-1} + \frac{2m+2k}{ns+ms+s+2}\right) (T(r, f) + T(r, g)) + S(r, f) + S(r, g)$$

which contradicts $(n-m-1)s \geq 2m+3$.

**Case 3:** If $F \equiv G$, that is

$$F^* = G^* + c \tag{19}$$

where $c$ is a constant, then it follows that

$$T(r,f) = T(r,g) + S(r,f). \tag{20}$$

Suppose that $c \neq 0$, by (2), (3), (8), (9), (20), the second fundamental theorem, and lemma 1, we have

$$\begin{aligned}
T(r,G^*) &\leq \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{G^*+c}\right) + \overline{N}\left(r,G^*\right) + S(r,g)\\
(n+m+1)T(r,g) &\leq \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{F^*}\right) + \overline{N}\left(r,G^*\right) + S(r,g)\\
&\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^m+\cdots+\frac{m+n+1}{n+1}\frac{a_0}{a_m}}\right) + \overline{N}\left(r,\frac{1}{f}\right)\\
&\quad + \overline{N}\left(r,\frac{1}{f^m+\cdots+\frac{m+n+1}{n+1}\frac{a_0}{a_m}}\right) + \overline{N}(r,g) + S(r,g)\\
&\leq \left(\frac{3}{s}+2m\right)T(r,f) + S(r,f) + S(r,g)
\end{aligned} \tag{21}$$

which contradicts our assumption $(n-m-1)s \geq 10$. Therefore $F^* = G^*$ that is,

$$f^{n+1}\left(\frac{a_m f^m}{m+n+1} + \frac{a_{m-1}f^{m-1}}{m+n} + \cdots + \frac{a_0}{n+1}\right) = g^{n+1}\left(\frac{a_m g^m}{m+n+1} + \frac{a_{m-1}g^{m-1}}{m+n} + \cdots + \frac{a_0}{n+1}\right). \tag{22}$$

Let $h = \frac{f}{g}$. If $h$ is a constant, then substituting $f = gh$ into (22) we deduce,

$$\frac{a_m g^{m+n+1}(h^{m+n+1}-1)}{m+n+1} + \frac{a_{m-1}g^{n+m}(h^{m+n}-1)}{m+n} + \cdots + \frac{a_0 g^{n+1}(h^{n+1}-1)}{n+1} = 0 \tag{23}$$

which implies $h^d = 1$, where $d = (n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1), a_{m-i} \neq 0$, for some $i = 0,1,2,\ldots,m$.

Thus $f = tg$ for a constant $t$, such that $t^d = 1$, where $d = (n+m+1, n+m, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$, for some $i = 0,1,\ldots,m$.

If $h$ is not a constant, then by (23) $f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$\begin{aligned}
R(\omega_1,\omega_2) = \omega_1^{n+1}&\left(\frac{a_m\omega_1^m}{n+m+1} + \frac{a_{m-1}\omega_1^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right)\\
&-\omega_2^{n+1}\left(\frac{a_m\omega_2^m}{n+m+1} + \frac{a_{m-1}\omega_2^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1}\right)
\end{aligned}$$

This completes the proof of Theorem 1.1.

### 3.2. Proof of Theorem 1.2.

Let

$$F=\frac{f^{n}P(f)f'}{z}, \qquad G=\frac{g^{n}P(g)g'}{z} \tag{24}$$

and

$$\begin{aligned} F^{*} &= \frac{a_m f^{m+n+1}}{m+n+1}+\frac{a_{m-1}f^{m+n}}{m+n}+\cdots+\frac{a_0 f^{n+1}}{n+1} \\ G^{*} &= \frac{a_m g^{m+n+1}}{m+n+1}+\frac{a_{m-1}g^{m+n}}{m+n}+\cdots+\frac{a_0 g^{n+1}}{n+1} \end{aligned} \tag{25}$$

Thus we obtain that $F$ and $G$ share 1 CM.

Since $(F^{*})'=Fz$, we deduce

$$m\left(r,\frac{1}{(F^{*})'}\right)\le m\left(r,\frac{1}{zF}\right)+S(r,f)\le m\left(r,\frac{1}{F}\right)+logr+S(r,f) \tag{26}$$

and by the first fundamental Theorem,

$$\begin{aligned} T(r,F^{*}) &\le T(r,F)+N\left(r,\frac{1}{F^{*}}\right)-N\left(r,\frac{1}{F}\right)+S(r,f) \\ &\le T(r,F)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-b_1}\right)+\cdots+N\left(r,\frac{1}{f-b_m}\right) \\ &\quad -N\left(r,\frac{1}{f-c_1}\right)-\cdots-N\left(r,\frac{1}{f-c_m}\right)-N\left(r,\frac{1}{f'}\right)+logr+S(r,f) \end{aligned} \tag{27}$$

where $b_1,b_2,\ldots,b_m$ are roots of the algebraic equation,

$$\frac{a_m z^{m}}{m+n+1}+\frac{a_{m-1}z^{m-1}}{m+n}+\cdots+\frac{a_0}{n+1}=0$$

and $c_1,c_2,\ldots,c_m$ are roots of the algebraic equation,

$$a_m z^{m}+a_{m-1}z^{m-1}+\cdots+a_0=0.$$

By Lemma 2, one of the following three cases holds.

**Case 1:**

$$T(r,F)\le N_2(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2(r,G)+N_2\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g). \tag{28}$$

On the other hand, we have

$$N_2(r,F)+N_2\left(r,\frac{1}{F}\right)\le 2\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-c_1}\right)+\cdots+N\left(r,\frac{1}{f-c_m}\right)+N\left(r,\frac{1}{f'}\right)+2logr \tag{29}$$

$$N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\leq 2\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-c_1}\right)+\cdots+N\left(r,\frac{1}{g-c_m}\right)+N\left(r,\frac{1}{g'}\right)+2logr \tag{30}$$

From (27) - (30), we obtain

$$T(r,F^*)\leq\left(\frac{4}{s}+m+1\right)T(r,f)+\left(\frac{5}{s}+m+1\right)T(r,g)+4logr+S(r,f)+S(r,g)$$

$$\left(n-\frac{4}{s}\right)T(r,f)\leq\left(\frac{5}{s}+m+1\right)T(r,g)+4logr+S(r,f)+S(r,g) \tag{31}$$

Similarly,

$$\left(n-\frac{4}{s}\right)T(r,g)\leq\left(\frac{5}{s}+m+1\right)T(r,f)+4logr+S(r,f)+S(r,g) \tag{32}$$

From (31) and (32), we deduce that $(n-m-1)s\leq 9$, which contradicts the assumption $(n-m-1)s\geq 10$

**Case 2:** Suppose $FG\equiv 1$, that is

$$f^nP(f)f'g^nP(g)g'\equiv z^2.$$

Proceeding as in the proof of Theorem 1.1(case 2), we get a contradiction.

**Case 3:** $F\equiv G$ that is $F^*=G^*+c$.

Proceeding as in the proof of Theorem 1.1(case 3), we get a conclusion of Theorem 1.2.

Therefore, we complete the proof of Theorem 1.2.

## 4. Consequences of Theorem 1.1 and Theorem 1.2

Under the condition of Theorem 1.1 and Theorem 1.2, setting $P(z)=(z-1)^m$, we get following results as immediate consequences of Theorem 1.1 and Theorem 1.2.

Consequences of Theorem 1.1

(A) Let $f$ and $g$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$.Let $n,m,s$ be positive integers with ,$(n-m-1)s\geq max\{10,2m+3\}$.If $f^nP(f)f'$ and $g^nP(g)g'$ share 1 CM , and

(1) if $s=1$,

(i) $m=1, n\geq 12$ then $g=\dfrac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f=\dfrac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$

(ii) $m=2, n\geq 13$ then $f\equiv g$

(iii) $m\geq 3$ then $f^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}f^{m-k}=g^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}g^{m-k}$

(2) if $s=2$,

(i) $m=1, n\geq 7$ then $g=\dfrac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f=\dfrac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$

(ii) $m=2, n\geq 8$ then $f\equiv g$

(iii) $m\geq 3$ then $f^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}f^{m-k}=g^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}g^{m-k}$

Similar observations can be made for $s=3,4,\ldots,9$

(3) if $s\geq 10$,

(i) $m=1, n\geq 2$ then $g=\dfrac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f=\dfrac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$

(ii) $m=2, n\geq 3$ then $f\equiv g$

(iii) $m\geq 3$ then $f^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}f^{m-k}=g^{n+1}\sum_{k=0}^{m}\dfrac{(-1)^k C_m^{m-k}}{n+m-k+1}g^{m-k}$.

Thus as per the above observations , we see that as multiplicity $'s'$ increases , the value of $n$ decreases.

Consequences of Theorem 1.2

(B) Let $f$ and $g$ be two trancendental meromorphic functions, whose zeros and poles are of multiplicities atleast $s$. Let $n,m,s$ be positive integers with $(n-m-1)s\geq max\{10,2m+3\}$. If $f^nP(f)f'$ and $g^nP(g)g'$ share $z$ CM, we observe results similar to consequences mentioned above.

Under the condition of Theorem 1.1 and Theorem 1.2, setting $P(z)=z^m-a$, $a\neq 0$ we get following results as immediate consequences of Theorem 1.1 and Theorem 1.2.

Consequences of Theorem 1.1

(C) Let $f$ and $g$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities atleast $s$. Let $n,m,s$ be positive integers with $(n-m-1)s\geq max\{10,2m+3\}$.If $f^nP(f)f'$ and $g^nP(g)g'$ share 1 CM , and

(1) if $s=1$,

(i) $m=1$,$n\geq 12$ then $g=\dfrac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f=\dfrac{ah(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$

(ii) $m \geq 2$ then $g = \left[\frac{a(n+m+1)(1-h^{n+1})}{(n+1)(1-h^{n+m+1})}\right]^{\frac{1}{m}}$, $f = \left[\frac{a(n+m+1)(1-h^{n+1})}{(n+1)(1-h^{n+m+1})}\right]^{\frac{1}{m}} h$.

(2) if $s \geq 10$,

(i) $m = 1, n \geq 2$ then $g = \frac{a(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f = \frac{ah(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$

(ii) $m \geq 2$ then $g = \left[\frac{a(n+m+1)(1-h^{n+1})}{(n+1)(1-h^{n+m+1})}\right]^{\frac{1}{m}}$, $f = \left[\frac{a(n+m+1)(1-h^{n+1})}{(n+1)(1-h^{n+m+1})}\right]^{\frac{1}{m}} h$

## 5. Final Remarks

It follows from the proof of Theorem 1.2 that if condition $f^n P(f) f'$ and $g^n P(g) g'$ share $z$ CM is replaced by the condition $f^n P(f) f'$ and $g^n P(g) g'$ share $\alpha(z)$ CM, where $\alpha$ is a meromorphic function such that $\alpha \neq 0, \infty$ and $T(r,\alpha) = o\{T(r,f), T(r,g)\}$, the conclusion of the Theorem 1.2 still holds.


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