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INEQUALITIES ON SEVERAL QUASI-ARITHMETIC MEANS

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Abstract. Inequalities on several quasi-arithmetic means are established by using convexity and concavity.

1. Introduction

In [3] Y. H. Kim proved the following

Theorem A. *Let* $a_i \ge 0, i = 1,2,\ldots,n,\ x \ge 1$ *and* $y \ge 0$. *Then*

$$\left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y} \le \left(\sum_{i=1}^{n}\frac{{a_i}^x}{n}\right)^{\frac{x+y}{x}} \le \sum_{i=1}^{n}\frac{{a_i}^{x+y}}{n}, \tag{1}$$

with all equalities holding if and only if all a_i *are the same.*

Let $a_i > 0, i = 1,2,\ldots,n,\ 0 < x \le 1$,*and* $y \ge 0$. *Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{x+y}{n}} \le \left(\sum_{i=1}^{n}\frac{{a_i}^x}{n}\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y}. \tag{2}$$

Theorem B. *Let* $a_i \ge 0, i = 1,2,\ldots,n,\ 0 < x \le 1$ *and* $-x \le y \le 0$. *Then*

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{x+y}{n}} \le \sum_{i=1}^{n}\frac{{a_i}^{x+y}}{n} \le \left(\sum_{i=1}^{n}\frac{{a_i}^x}{n}\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y}, \tag{3}$$

with all equalities holding if and only if all a_i *are the same.*

Theorem C.(corrected) *Let* $a_i > 0,\ i = 1,2,\ldots,n,\ x \ge 1$,*and* $-x \ge y \ge -2x$. *Then*

$$\left(\sum_{i=1}^{n}\frac{a_i^x}{n}\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y} \le \left(\prod_{i=1}^{n} a_i\right)^{\frac{x+y}{n}} \le \sum_{i=1}^{n}\frac{{a_i}^{x+y}}{n}. \tag{4}$$

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Theorem D.(corrected) *Let* $a_i > 0$, $i = 1,2,\ldots,n$, $x \ge 1$,*and* $-2x \ge y$. *Then*

$$\left(\sum_{i=1}^{n}\frac{a_i^x}{n}\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y} \le \left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \le \left(\sum_{i=1}^{n}\frac{1}{n{a_i}^x}\right)^{\frac{-(x+y)}{x}} \le \sum_{i=1}^{n}\frac{{a_i}^{x+y}}{n}. \tag{5}$$

In [1] S. Abramovich, J. Pečarić and S. Varošance have proved several generalizations of Kim's results, one of which is the following

Theorem E. *Let* $x \ge 1$, $y \ge 0$, *and* $\frac{x+y}{x} \ge x$. *If* $a_i \ge 0$, $i = 1,2,\ldots,n$, *then*

$$\left(\sum_{i=1}^{n}\frac{a_i}{n}\right)^{x+y} \le \left(\sum_{i=1}^{n}\frac{a_i^x}{n}\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n}\frac{a_i^{\frac{x+y}{x}}}{n}\right)^{x} \le \sum_{i=1}^{n}\frac{{a_i}^{x+y}}{n}. \tag{6}$$

Inequalities (6) follow also from Theorem C and Corollary 4 in [1].

In this paper we give some generalizations of the above inequalities by using convexity and concavity.

2. Main results

Throughout,let $f, g : [a,b] \to [c,d]$, $a_i \in [c,d]$, $0 \le \alpha_i \le 1$, $i = 1,2,\ldots,n$ and $\sum_{i=1}^{n}\alpha_i = 1$.

Lemma 1. *If one of the following conditions holds*

(i) g *is strictly increasing,* $g^{-1}\circ f$ *is convex and strictly monotone on* $[a,b]$;

(ii) g *is strictly decreasing,* $g^{-1}\circ f$ *is concave and strictly monotone on* $[a,b]$

then

$$f\left(\sum_{i=1}^{n}\alpha_i f^{-1}(a_i)\right) \le g\left(\sum_{i=1}^{n}\alpha_i g^{-1}(a_i)\right). \tag{7}$$

If one of the following conditions holds

(iii) g *is strictly increasing,* $g^{-1}\circ f$ *is concave and strictly monotone on* $[a,b]$;

(iv) g *is strictly decreasing,* $g^{-1}\circ f$ *is convex and strictly monotone on* $[a,b]$

then

$$g\left(\sum_{i=1}^{n}\alpha_i g^{-1}(a_i)\right) \le f\left(\sum_{i=1}^{n}\alpha_i f^{-1}(a_i)\right). \tag{8}$$

Proof. We will prove the result for case (i), where $g^{-1}\circ f$ is strictly increasing on $[a,b]$, a similar argument establishes the results for other cases. If $x, y \in [a,b]$ and $x < y$,since $g^{-1}\circ f$ is strictly increasing on $[a,b]$,we have $g^{-1}(f(x)) = (g^{-1}\circ f)(x) < (g^{-1}\circ f)(y) = g^{-1}(f(y))$.

Since g is strictly increasing,it follows that $f(x) < f(y)$, so that f is strictly increasing on $[a,b]$ and f^{-1}exists.

Since $g^{-1}\circ f$ is convex on $[a,b]$, we have

$$(g^{-1}\circ f)\left(\sum_{i=1}^{n}\alpha_i f^{-1}(a_i)\right)\le\sum_{i=1}^{n}\alpha_i\left(g^{-1}\circ f\right)(f^{-1}(a_i))=\sum_{i=1}^{n}\alpha_i g^{-1}((f\circ f^{-1})(a_i))=\sum_{i=1}^{n}\alpha_i g^{-1}(a_i)$$

Since g is strictly increasing,it follows from the above result that

$$f\left(\sum_{i=1}^{n}\alpha_i f^{-1}(a_i)\right)\le g\left(\sum_{i=1}^{n}\alpha_i g^{-1}(a_i)\right)$$

This completes the proof.

Proposition 1. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $x+y\ge 0, x\ge 1$ *or* $x<0, x+y\le 0$ *or* $0<x\le 1, x+y\le 0$, *then*

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^{x+y}\le\left(\sum_{i=1}^{n}\alpha_i a_i^{x}\right)^{\frac{x+y}{x}};\tag{9}$$

(ii) *If* $x+y\ge 0, 0<x\le 1$ *or* $x<0, x+y\ge 0$ *or* $x\ge 1, x+y\le 0$, *then*

$$\left(\sum_{i=1}^{n}\alpha_i a_i^{x}\right)^{\frac{x+y}{x}}\le\left(\sum_{i=1}^{n}\alpha_i a_i\right)^{x+y}.\tag{9'}$$

Proof. Let $f(t)=t^{x+y}$ and $g(t)=t^{\frac{x+y}{x}}$. Then (9) follows from (7) and (9$'$) follows from (8).

Proposition 2. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $xy\ge 0, x\ne 0$ *or* $x>0, y\le 0, x+y\le 0$ *or* $x<0, y\ge 0, x+y\ge 0$, *then*

$$\left(\sum_{i=1}^{n}\alpha_i a_i^{x}\right)^{\frac{x+y}{x}}\le\sum_{i=1}^{n}\alpha_i a_i^{x+y};\tag{10}$$

(ii) *If* $x>0, y\le 0, x+y\ge 0$ *or* $x<0, y\ge 0, x+y\le 0$, *then*

$$\left(\sum_{i=1}^{n}\alpha_i a_i^{x}\right)^{\frac{x+y}{x}}\ge\sum_{i=1}^{n}\alpha_i a_i^{x+y}.\tag{10'}$$

Proof. Let $f(t)=t^{\frac{x+y}{x}}$ and $g(t)=t$. Then (10) follows from (7) and (10$'$)follows from (8).

Remark 1. Suppose $a_i>0,\ i=1,2,\dots,n$, it follows from (9) and (10) that, if $x\ge 1, y\ge 0$ or $x<0, y\le 0$, or $0<x\le 1, x+y\le 0$, then

$$\left(\sum_{i=1}^{n}\alpha_i a_i\right)^{x+y}\le\left(\sum_{i=1}^{n}\alpha_i a_i^{x}\right)^{\frac{x+y}{x}}\le\sum_{i=1}^{n}\alpha_i a_i^{x+y},$$

which extends the inequality (1) in Theorem A with $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$.

Proposition 3. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

$$\prod_{i=1}^{n} a_i^{\alpha_i x} \le \sum_{i=1}^{n} \alpha_i a_i^x,\ \forall x. \tag{11}$$

Proof. Let $f(t) = t^{\frac{1}{x}}$ and $g(t) = e^t$. Then (11) follows from (8).

We note that the inequality (11) reduces to

$$\text{geometric mean} \le \text{arithmetic mean},$$

if we choose $x = 1$ and reduces to

$$\text{harmonic mean} \le \text{geometric mean},$$

if we choose $x = -1$.

Remark 2. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from $(9^{'})$,$(10^{'})$ and (11) that, if $0 < x \le 1, y \le 0, x + y \ge 0$, then

$$\prod_{i=1}^{n} a_i^{\alpha_i (x+y)} \le \sum_{i=1}^{n} \alpha_i a_i^{x+y} \le \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y},$$

which is the inequality (3) in Theorem B with $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$.

Proposition 4. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $\frac{x+y}{x} \le 0$, *then*

$$\prod_{i=1}^{n} a_i^{\alpha_i (x+y)} \ge \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}}; \tag{12}$$

(ii) *If* $\frac{x+y}{x} \ge 0$, *then*

$$\prod_{i=1}^{n} a_i^{\alpha_i (x+y)} \le \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}}. \tag{$12^{'}$}$$

Proof. Let $f(t) = t^{\frac{x+y}{x}}$ and $g(t) = e^t$. Then (12) follows from (7) and $(12^{'})$ follows from (8).

Remark 3. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from $(9^{'})$ and $(12^{'})$ that, if $0 < x \le 1, x + y \ge 0$, then

$$\prod_{i=1}^{n} a_i^{\alpha_i (x+y)} \le \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y},$$

which extends the inequality (2) in Theorem A with $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$.

Proposition 5. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $x \le 0$, *then*

$$\prod_{i=1}^{n} a_i^{\alpha_i x} \ge \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x}; \tag{13}$$

(ii) *If* $x \ge 0$, *then*

$$\prod_{i=1}^{n} a_i^{\alpha_i x} \le \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x}. \tag{13$'$}$$

Proof. Let $f(t) = t^x$ and $g(t) = e^t$. Then (13) follows from (7) and ($13^{'}$) follows from (8).

Remark 4. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from ($9^{'}$),(11) and (13) that, if $x \ge 1, x+y \le 0$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \le \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \le \prod_{i=1}^{n} a_i^{\alpha_i(x+y)} \le \sum_{i=1}^{n} \alpha_i a_i^{x+y},$$

which extends the inequality (4) in Theorem C with $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$.

Remark 5. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from (12) and ($13^{'}$) that, if $x < 0, x+y \ge 0$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \le \prod_{i=1}^{n} a_i^{\alpha_i(x+y)} \le \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y}.$$

Proposition 6. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $\frac{x+y}{x} \ge 0$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{\frac{-(x+y)}{x}} \le \prod_{i=1}^{n} a_i^{\alpha_i(x+y)}; \tag{14}$$

(ii) *If* $\frac{x+y}{x} \le 0$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{\frac{-(x+y)}{x}} \ge \prod_{i=1}^{n} a_i^{\alpha_i(x+y)}. \tag{14$'$}$$

Proof. Let $f(t) = t^{\frac{-(x+y)}{x}}$ and $g(t) = e^t$. Then (14) follows from (7) and ($14^{'}$) follows from (8).

Remark 6. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from (9),(13) and (14) that , if $x \geq 1, x+y \geq 0$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{\frac{-(x+y)}{x}} \leq \prod_{i=1}^{n} a_i^{\alpha_i(x+y)} \leq \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^{x}\right)^{\frac{x+y}{x}}.$$

Proposition 7. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then* $\frac{x+y}{x} \leq -1$ *or* $\frac{x+y}{x} \geq 0$, then

$$\sum_{i=1}^{n} \alpha_i a_i^{x+y} \geq \left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{\frac{-(x+y)}{x}}. \tag{15}$$

Proof. Let $f(t) = t^{\frac{-(x+y)}{x}}$ and $g(t) = t$. Then (15) follows from (7).

Remark 7. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from $(9')$,$(13')$,$(14')$ and (15) that, if $x \geq 1, \frac{x+y}{x} \leq -1$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{x}\right)^{\frac{x+y}{x}} \leq \left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \leq \prod_{i=1}^{n} a_i^{\alpha_i(x+y)} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{\frac{-(x+y)}{x}} \leq \sum_{i=1}^{n} \alpha_i a_i^{x+y},$$

which is the inequality (5) in Theorem D with $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$.

Proposition 8. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $x \geq 1$ *or* $x < 0$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x} \leq \sum_{i=1}^{n} \alpha_i a_i^{x+y}; \tag{16}$$

(ii) *If* $0 < x \leq 1$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x} \geq \sum_{i=1}^{n} \alpha_i a_i^{x+y}. \tag{16$'$}$$

Proof. Let $f(t) = t^x$ and $g(t) = t$. Then (16) follows from (7) and $(16')$ follows from (8).

Proposition 9. *Suppose* $a_i > 0,\ i = 1,2,\dots,n$. *Then*

(i) *If* $x > 0, x+y \geq x^2$ *or* $x > 0, x+y \leq 0$ *or* $x < 0 \leq x+y \leq x^2$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{x}\right)^{\frac{x+y}{x}} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x}; \tag{17}$$

(ii) *If* $x > 0, 0 \leq x+y \leq x^2$ *or* $x < 0, x+y \geq x^2$ *or* $x < 0, x+y \leq 0$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^{x}\right)^{\frac{x+y}{x}} \geq \left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x}. \tag{17$'$}$$

Proof. Let $f(t) = t^{\frac{x+y}{x}}$ and $g(t) = t^x$. Then (17) follows from (7) and ($17'$) follows from (8).

Remark 8. Suppose $a_i \geq 0,\ i = 1,2,\dots,n$, it follows from (9), (16) and (17), that if $x \geq 1, x+y \geq x^2$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x} \leq \sum_{i=1}^{n} \alpha_i a_i^{x+y}.$$

These last inequalities follow also from Corollary 4 in [1] where general results are stated. The special case $\alpha_i = \frac{1}{n}, i = 1,2,\dots,n$ is stated there as Corollary 6.

Remark 9. Suppose $a_i > 0,\ i = 1,2,\dots,n$, it follows from ($9'$), ($16'$) and ($17'$), that if $0 < x \leq 1, 0 \leq x+y \leq x^2$, then

$$\left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \geq \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \geq \left(\sum_{i=1}^{n} \alpha_i a_i^{\frac{x+y}{x}}\right)^{x} \geq \sum_{i=1}^{n} \alpha_i a_i^{x+y}.$$

Proposition 10. *If we choosing* $f(t) = t^{\frac{1}{x}}, g(t) = t^{\frac{1}{y}}, t > 0, x \neq 0, y \neq 0$, *then it follows from* (7) *that if* $0 < x \leq y$ *or* $x < 0 < y$ *or* $x \leq y < 0$, $a_i > 0, i = 1,2,\dots,n$, *then*

$$\left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{1}{x}} \leq \left(\sum_{i=1}^{n} \alpha_i a_i^y\right)^{\frac{1}{y}}, \tag{18}$$

which shows that the mean of order t, *defined by* $M_t(a,\alpha) = (\sum_{i=1}^{n} \alpha_i a_i^t)^{\frac{1}{t}}$ *is a nondecreasing function of* t *for* $-\infty < t < \infty$ (see [2, p17] or [4, p15]).

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