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(θ,ϕ)-DERIVATIONS AS HOMOMORPHISMS OR AS ANTI-HOMOMORPHISMS ON A NEAR RING

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Abstract. Let N be a near ring. An additive mapping $d : N \longrightarrow N$ is said to be a (θ,ϕ)-derivation on N if there exist mappings $\theta,\phi : N \longrightarrow N$ such that $d(xy) = \theta(x)d(y) + d(x)\phi(y)$ holds for all $x, y \in N$. In the context of 3-prime and 3-semiprime nearrings, we show that for suitably-restricted θ and ϕ, there exist no nonzero (θ,ϕ)-derivations which act as a homomorphism or an anti-homomorphism on N or a nonzero semigroup ideal of N.

1. Introduction

Throughout the paper N will be a zero-symmetric right nearring with multiplicative centre Z; and for arbitrary $x, y \in N$, $[x, y]$ will denote the commutator $xy - yx$. The nearring N is said to be 3-prime if $xNy = \{0\}$, for $x, y \in N$ implies that $x = 0$ or $y = 0$ and 3-semiprime if $xNx = \{0\}$ implies that $x = 0$; moreover, N is called reduced if N contains no nonzero nilpotents. A nonempty subset A of N is called a semigroup ideal if $AN \subseteq A$ and $NA \subseteq A$.

An additive mapping $d : N \longrightarrow N$ is said to be a derivation on N if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$. Or equivalently (cf [5]) $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. As in [1], an additive mapping $d : N \longrightarrow N$ is called a (θ,ϕ)-derivation if there exist mappings $\theta,\phi : N \longrightarrow N$ such that $d(xy) = \theta(x)d(y) + d(x)\phi(y)$ for all $x, y \in N$. If I denotes the identity map on N, an (I, I)-derivation is just a derivation, so the notion of (θ,ϕ)-derivation generalizes that of derivation. Moreover, the generalization is not trivial, as the following example shows:

Example 1. Let N_1 be an arbitrary zero-symmetric nearring; and N_2 be an algebra over a field of characteristic not 2, with basis $\{u, v, w\}$ and multiplication given by $uv = w$ and all other two-products of basis elemants equal to 0. Let $N = N_1 \oplus N_2$. Define $\theta,\phi : N \longrightarrow N$ by $\theta((x_1, x_2)) = (x_1, 0)$ and $\phi((x_1, x_2)) = (x_1^2, 0)$; and let $d : N \longrightarrow N$ be given by $d((x_1, x_2)) = (0, h(x_2))$, where $h : N_2 \longrightarrow N_2$ is the additive map such that $h(u) = u$, $h(v) = v$, and $h(w) = 0$. It is easy to verify that d is a (θ,ϕ)-derivation which is not a derivation.

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In [4] it is proved that if R is a semiprime ring and d is a derivation on R which is either an endomorphism or an anti-endomorphism on R, then $d = 0$. Of course derivations which are not endomorphisms or anti-endomorphisms on R may behave as such on certain subsets of R; for example, any derivation d behaves as the zero endomorphism on the subring C consisting of all constants (i.e. elements x for which $d(x) = 0$.) In fact, in a semiprime ring, d may behave as an endomorphism on a proper ideal of R. However, as noted in [4], the behaviour of d is somewhat restricted in the case of a prime ring.

Recently the authors in [2] considered (θ,ϕ)-derivations d acting as homomorphisms or anti-homomorphisms on a nonzero Lie ideal of a prime ring such that $u^2 \in U$ for all $u \in U$ and concluded that $d = 0$. The purpose of the present paper is to establish similar results in the setting of semigroup ideal of 3-prime and 3-semiprime nearrings admitting a (θ,ϕ)-derivation.

Usually, we shall require that $\theta(N) = N$, and that $\theta(ab) = \theta(a)\theta(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in N$. A (θ,ϕ)-derivation with θ and ϕ satisfying these conditions will be called a standard (θ,ϕ)-derivation.

2. Preliminary results

We begin with the following Lemma that was proved by the second author in [3].

Lemma 2.1. *Let N be a 3-prime nearring and A be a nonzero semigroup ideal of N.*

(i) *If $x \in N$ and $xA = \{0\}$ or $Ax = \{0\}$, then $x = 0$.*
(ii) *If $x, y \in N$ and $xAy = \{0\}$, then $x = 0$ or $y = 0$.*
(iii) *If $x \in N$ and x centralizes A, then $x \in Z$.*

Lemma 2.2. *Let N be an arbitrary nearring. If d is a standard (θ,ϕ)-derivation on N, then N satisfies the following partial distributive law:*

$$a(\theta(b)d(c) + d(b)\phi(c)) = a\theta(b)d(c) + ad(b)\phi(c) \text{ for all } a, b, c \in N.$$

Proof. For $a, b, c \in N$, we have

$$\begin{aligned} d((ab)c) &= \theta(ab)d(c) + d(ab)\phi(c) \\ &= \theta(a)\theta(b)d(c) + (\theta(a)d(b) + d(a)\phi(b))\phi(c) \\ &= \theta(a)\theta(b)d(c) + \theta(a)d(b)\phi(c) + d(a)\phi(b)\phi(c) \quad \text{for all } a, b, c \in N. \end{aligned}$$

Again

$$\begin{aligned} d(a(bc)) &= \theta(a)d(bc) + d(a)\phi(bc) \\ &= \theta(a)(\theta(b)d(c) + d(b)\phi(c)) + d(a)\phi(b)\phi(c) \quad \text{for all } a, b, c \in N. \end{aligned}$$

Comparing both expressions, we obtain

$$\theta(a)(\theta(b)d(c)+d(b)\phi(c))=\theta(a)\theta(b)d(c)+\theta(a)d(b)\phi(c) \text{ for all } a,b,c\in N.$$

Since $\theta(N)=N$, it follows that

$$a(\theta(b)d(c)+d(b)\phi(c))=a\theta(b)d(c)+ad(b)\phi(c) \text{ for all } a,b,c\in N.$$ □

Lemma 2.3. *Let N be an arbitrary zero-symmetric nearring. If d is a (θ,ϕ)-derivation on N and $\theta(2a)=2\theta(a)$ for all $a\in N$, then $d(ab)=d(a)\phi(b)+\theta(a)d(b)$ for all $a,b\in N$.*

Proof.

$$\begin{aligned} d((a+a)b) &= \theta(a+a)d(b)+d(a+a)\phi(b)\\ &= (\theta(a)+\theta(a))d(b)+(d(a)+d(a))\phi(b)\\ &\quad \theta(a)d(b)+\theta(a)d(b)+d(a)\phi(b)+d(a)\phi(b)\end{aligned}$$

and

$$d(ab+ab)=\theta(a)d(b)+d(a)\phi(b)+\theta(a)d(b)+d(a)\phi(b).$$

Equating the two expressions yields the desired result. □

The proof of the next lemma is trivial.

Lemma 2.4. *Let N be a reduced nearring.*

(i) *If $a,b\in N$ and $ab=0$, then $ba=0$.*
(ii) *If $a,b\in N$ and $ab=0$, then $aNb=\{0\}$.*

Lemma 2.5. *Let N be an arbitrary nearring. If A is a nonzero semigroup ideal of N and d is a standard (θ,ϕ)-derivation on N which acts as homomorphism on A, then*

$$(d(z)-\theta(z))\theta(x)d(y)=0 \quad \textit{for all } x,y,z\in A.$$

Proof. Since d acts as a homomorphism on A, we have

$$d(xy)=\theta(x)d(y)+d(x)\phi(y)=d(x)d(y) \quad \text{for all } x,y\in A. \tag{2.1}$$

Replacing x by zx in (2.1), we get

$$\theta(z)\theta(x)d(y)+d(z)d(x)\phi(y)=d(z)d(x)d(y) \quad \text{for all } x,y,z\in A.$$

and it follows that

$$\theta(z)\theta(x)d(y)+d(z)d(x)\phi(y)=d(z)(\theta(x)d(y)+d(x)\phi(y)) \quad \text{for all } x,y,z\in A.$$

Using Lemma 2.2, we find that

$$\theta(z)\theta(x)d(y)+d(z)d(x)\phi(y)=d(z)\theta(x)d(y)+d(z)d(x)\phi(y) \quad \text{for all } x,y,z\in A.$$

i.e.

$$\theta(z)\theta(x)d(y)=d(z)\theta(x)d(y) \quad \text{for all } x,y,z\in A.$$

Thus,

$$(d(z)-\theta(z))\theta(x)d(y)=0 \quad \text{for all } x,y,z\in A.$$ □

Lemma 2.6. *Let N be an arbitrary nearring. If A is a nonzero semigroup ideal of N and d is a standard (θ,ϕ)-derivation on N which acts as an anti-homomorphism on A, then*

$$[\theta(z),d(y)]\theta(x)d(y)=0 \quad \textit{for all } x,y,z\in A.$$

Proof. Let d act as an anti-homomorphism on A. Then we have

$$d(xy)=\theta(x)d(y)+d(x)\phi(y)=d(y)d(x) \quad \text{for all } x,y\in A. \tag{2.2}$$

Replacing x by xy in (2.2), we obtain

$$\theta(x)\theta(y)d(y)+d(y)d(x)\phi(y)=d(y)d(xy) \quad \text{for all } x,y\in A--$$

i.e.

$$\theta(x)\theta(y)d(y)+d(y)d(x)\phi(y)=d(y)(\theta(x)d(y)+d(x)\phi(y)) \quad \text{for all } x,y\in A.$$

Using Lemma 2.2, we get

$$\theta(x)\theta(y)d(y)+d(y)d(x)\phi(y)=d(y)\theta(x)d(y)+d(y)d(x)\phi(y) \quad \text{for all } x,y\in A,$$

so that

$$\theta(x)\theta(y)d(y)=d(y)\theta(x)d(y) \text{ for all } x,y\in A. \tag{2.3}$$

Replacing x by zx in (2.3), we find that

$$\theta(z)\theta(x)\theta(y)d(y)=d(y)\theta(z)\theta(x)d(y) \quad \text{for all } x,y,z\in A.$$

Again using (2.3), the above relation gives that

$$\theta(z)d(y)\theta(x)d(y)=d(y)\theta(z)\theta(x)d(y) \quad \text{for all } x,y,z\in A.$$

$$(\theta(z)d(y)-d(y)\theta(z))\theta(x)d(y)=0 \quad \text{for all } x,y,z\in A--$$

i.e.

$$[\theta(z),d(y)]\theta(x)d(y)=0 \quad \text{for all } x,y,z\in A.$$ □

3. Main results

Theorem 3.1. *Let N be 3-prime. Let A be a nonzero semigroup ideal of N such that $\theta(A) = A$ and $\phi(A) = A$. If d is a standard (θ,ϕ)-derivation on N which acts as a homomorphism or an anti-homomorphism on A, then $d = 0$.*

Proof. Assume first that d act as a homomorphism on A. By Lemma 2.5, we have $(d(z)-\theta(z))\theta(x)d(y) = 0$ for all $x, y, z \in A$; and since $\theta(A) = A$, we see that $(d(z)-\theta(z))Ad(y) = 0$ for all $y, z \in A$. It follows by Lemma 2.1 (ii) that $d(z)-\theta(z) = 0$ for all $z \in A$ or $d(A) = \{0\}$. In the latter case, using the definition of (θ,ϕ)-derivation and Lemma 2.1 (i), we get $d = 0$. Thus, assume $d(z)-\theta(z) = 0$ for all $z \in A$, in which case equation (2.1) implies $d(x)\phi(y) = 0$ for all $x, y \in A$. Since $\phi(A) = A$, another appeal to Lemma 2.1 (i) gives $d(A) = \{0\}$ and hence $d = 0$. ☐

Assume now that d acts as an anti-homomorphism on A. Since $\theta(A) = A$, Lemma 2.6 shows that $d(A)$ centralizes A; hence by Lemma 2.1 (iii) $d(A) \subseteq Z$. Thus, d acts as a homomorphism.

We proceed now to the case of 3-semiprime nearrings.

Theorem 3.2. *If N is 3-semiprime and d is a standard (θ,θ)-derivation which acts as a homomorphism on N, then $d = 0$.*

Proof. Using the same method in Lemma 2.5, we get

$$(d(z)-\theta(z))\theta(x)d(y) = 0, \text{ for all } x, y, z \in N.$$

Replacing z by zw, $w \in N$ in this equation, we have

$$\begin{aligned} 0 &= (d(zw)-\theta(zw))\theta(x)d(y) = (\theta(z)d(w)+d(z)\theta(w)-\theta(z)\theta(w))\theta(x)d(y) \\ &= (\theta(z)d(w)+(d(z)-\theta(z))\theta(w))\theta(x)d(y) \\ &= \theta(z)d(w)\theta(x)d(y)+(d(z)-\theta(z))\theta(wx)d(y) \\ &= \theta(z)d(w)\theta(x)d(y) \quad \text{for all } x, y, z, w \in N. \end{aligned}$$

Since $\theta(N) = N$ and $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in N$, we conclude that $\theta(x)d(y)N\theta(x)d(y) = 0$ for all $x, y \in N$, so that $Nd(y) = 0$ for all $y \in N$. Since N is 3-semiprime, we have $d = 0$. ☐

For reduced nearrings, which form a subclass of the class of 3-semiprime nearrings, we can get a result similar to Theorem 3.1 without the assumption that $\theta = \phi$.

Theorem 3.3. *Let N be reduced, and let d be a standard (θ,ϕ)-derivation. Assume also that $\phi(N) = N$ and $\theta(2a) = 2\theta(a)$ for all $a \in N$. If d acts as a homomorphism or an anti-homomorphism on N, then $d = 0$.*

Proof. Consider first d acting as a homomorphism on N. In a right nearring, $-xy = (-x)y$ for all $x, y \in N$; hence Lemma 2.5 gives

$$(d(z)-\theta(z))\theta(x)d(y) = 0 = (\theta(z)-d(z))\theta(x)d(y) \text{ for all } x, y \in N. \tag{3.1}$$

Replacing z by zw in the first equality and applying Lemma 2.3, we see that the left annihilator of $\theta(N)d(N)$ contains

$$\begin{aligned} d(z)\phi(w)+\theta(z)d(w)-\theta(z)\theta(w) &= d(z)\phi(w)+(\theta(z)-d(z)+d(z))d(w)-\theta(z)\theta(w) \\ &= d(z)\phi(w)+(\theta(z)-d(z))d(w)+d(zw)-\theta(zw). \end{aligned}$$

By (3.1) and Lemma 2.4 (ii), it follows that

$$d(z)\phi(w)\theta(x)d(y) = 0 \text{ for all } x, y, z, w \in N.$$

Postmultiplying by $\phi(u)\theta(v)$ and using the hypothesis that N is reduced, we see that $d(N)NN = \{0\}$; hence $d(N)d(N)d(N) = \{0\}$ and therefore $d(N) = \{0\}$. □

Finally, suppose that d acts as an anti-homomorphisms on N. By Lemma 2.6 we have

$$[\theta(z), d(y)]\theta(x)d(y) = 0 \text{ for all } x, y, z \in N.$$

so by Lemma 2.4

$$d(y)[\theta(z), d(y)]\theta(x) = 0 = \theta(z)d(y)[\theta(z), d(y)]\theta(x) = d(y)\theta(z)[\theta(z), d(y)]\theta(x).$$

Since $\theta(N) = N$, we conclude that $[\theta(z), d(y)]^2 = 0$ for all $y, z \in N$, so that $d(N) \subseteq Z$ and therefore d acts as a homomorphism.

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