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UNICITY OF MEROMORPHIC FUNCTIONS CONCERNING DIFFERENTIAL EQUATION

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Abstract. In this paper,we investigate the uniqueness of meromorphic function satisfying a differential equation. Our result improves some known results.

1. Introduction

In this paper, the term 'meromorphic' means meromorphic in the whole complex plane.It is assumed that the reader is familiar with notations of Nevanlinna Theory as in [2].We denote by $S(r,f)$ any function satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$, possibly outside the set of finite measure.We say $a(z)$ is a small meromorphic function of f if $T(r,a(z))=S(r,f)$.

Subhas. S. Bhoosnurmath and K. S. L. N. Prasad [1] have proved the following Theorem on uniqueness of meromorphic functions sharing a small meromorphic function.

Theorem A. *Let f be a non-constant transcendental meromorphic function with $N(r,f)+N\left(r,\frac{1}{f}\right)=S(r,f)$ Let $'a'$ be a small meromorphic function of f i.e.*

$$T(r,a)=S(r,f).$$

If f satisfies the equation,

$$kf'-f-(k-1)a=0$$

for $k\neq 0$, then $f=f'$.

2. We require the following definitions

By a *Monomial in f* we mean an expression of the type

$$M_j(f)=a_j(z)(f(z))^{n_{oj}}(f'(z))^{n_{1j}}\cdots(f^{(k)}(z))^{n_{kj}}$$

where $n_{0j},n_{1j},\ldots,n_{kj}$ are non-negative integers. We define $\bar{d}(M_j)=\sum_{i=o}^{k}n_{ij}$ as the *degree* of $M_j(f)$ and $\Gamma(M_j)=\sum_{i=o}^{k}(i+1)n_{ij}$ as the *weight* of $M_j(f)$.

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Differential polynomial in f is a finite sum of such monomials i.e.

$$P(f) = \sum_{j=1}^{s} a_j(z) M_j(f).$$

We define $\bar{d}(P) = max_{1 \le j \le s} \left\{ \bar{d}(M_j) \right\}$ as the *degree*, $\underline{d}(P) = min_{1 \le j \le s} \left\{ \bar{d}(M_j) \right\}$ as the *lower degree* and $\Gamma(P) = max_{1 \le j \le s} \left\{ \Gamma(M_j) \right\}$ as the *weight* of $P(f)$.

If $\bar{d}(P) = \underline{d}(P) = n$, $P(f)$ is a called *homogeneous differential* polynomial and *inhomogeneous* otherwise.

We shall improve Theorem A for a homogeneous differential polynomial in f as follows.

3. Statement of main Theorem

Theorem 1. *Let $f(z)$ be a non-constant transcendental meromorphic function with $N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f)$. Let $P(f)$ be a homogeneous differential polynomial in f.Let $'a'$ be a small meromorphic function of f i.e.*

$$T(r, a(z)) = S(r, f).$$

If f satisfies the equation

$$kP(f) - f - (k-1)\,a = 0 \quad for \quad k \ne 0 \tag{1}$$

Then, $f \equiv P(f)$.

We require the following lemmas to prove our results.

Lemma 1.([3]) *Let f_1 and f_2 be two non-constant meromorphic functions and $\alpha_1 \ne o$, $\alpha_2 \ne o$ be two small meromorphic functions satisfying,*

$$T(r, \alpha_i) = S(r, f) \;\; for \quad i = 1, 2$$

where,

$$T(r, f) = max\left\{T(r, f_1), T(r, f_2)\right\}.$$

If $\alpha_1 f_1 + \alpha_2 f_2 \equiv 1$, then

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}(r, \frac{1}{f_2} + \overline{N}(r, f_1) + o\left\{T(r, f)\right\},$$

$$T(r, f_2) < \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}(r, f_2) + o\left\{T(r, f)\right\}.$$

Lemma 2. *If f is a non-constant transcendental meromorphic function and if $P(f)$ is a homogeneous differential polynomial in f of degree n, then*

$$N\left(r, \frac{P(f)}{f^n}\right) \leq mn\left[\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r,f),$$

where m is the order of highest order derivative occuring in $P(f)$.

Proof of Lemma 2. Since m is the order of the highest derivative $f^{(m)}$ occuring in P. Then clearly a zero or a pole of f which is not a pole of any co-efficient $a(z)$ of P, is a pole of $\frac{P}{f^n}$ of degree mn atmost. Hence we have,

$$N(r, \frac{P}{f^n}) \leq mn\left[\overline{N}(r,f) + \overline{N}(r, \frac{1}{f})\right] + S(r,f).$$

Lemma 3.([4]) *If f is a non-constant trancsendental meromorphic function and if $P(f)$ is a homogeneous differential polynomial in f of degree n,then*

$$m\left(r, \frac{P(f)}{f^n}\right) = S(r,f).$$

Lemma 4. *If f is a non-constant transcendental meromorphic function and if $P(f)$ is a homogeneous differential polynomial in f of degree n then,*

$$N\left(r, \frac{1}{P(f)}\right) \leq mn\left[\overline{N}(r,f) + \overline{N}(r, \frac{1}{f})\right] + nN(r, \frac{1}{f}) + S(r,f),$$

where m is the order of highest order derivative occuring in $P(f)$.

Proof of Lemma 4. Consider,

$$\begin{aligned}
N\left(r, \frac{1}{P(f)}\right) &\leq N\left(r, \frac{f^n}{P(f)}\right) + N\left(r, \frac{1}{f^n}\right) \\
&\leq T\left(r, \frac{f^n}{P(f)}\right) + nN\left(r, \frac{1}{f}\right) + S(r,f) \\
&= m\left(r, \frac{P(f)}{f^n}\right) + N\left(r, \frac{P(f)}{f^n}\right) + nN(r, \frac{1}{f}) + O(1) \quad \textit{by Lemma 2 and 4}, \\
&\leq S(r,f) + mn\left[\overline{N}(r,f) + \overline{N}(r, \frac{1}{f})\right] + nN(r, \frac{1}{f}) + O(1) \\
N\left(r, \frac{1}{P(f)}\right) &\leq mn\left[\overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + nN\left(r, \frac{1}{f}\right) + S(r,f).
\end{aligned}$$

Proof of Theorem 1. We have,

$$\begin{aligned}
kP(f) - f - (k-1)\,a = 0 \\
\Rightarrow k\,(P(f) - a) = f - a \\
\Rightarrow \frac{f-a}{P(f)-a} = k \quad \text{where} \quad k \neq 0
\end{aligned}$$

Put $f_1 = \frac{1}{a}f$, $f_2 = k$, $f_3 = \frac{-k}{a}P(f)$ where $(a \neq 0)$, so that $f_1 + f_2 + f_3 \equiv 1$.

If $k \neq 1$, we get,

$$\frac{1}{a\,(1-k)}f - \frac{k}{a\,(1-k)}P(f) \equiv 1.$$

By lemma 1. we have,

$$T\,(r,f) < \overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\,(r,f) + S\,(r,f)\,, \tag{2}$$

$$T\,(r,P(f)) < \overline{N}\left(r,\frac{1}{P(f)}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\,(r,P(f)) + S\,(r,f)\,. \tag{3}$$

Also we have,

$$N\,(r,P(f)) \leq nN\,(r,f) + mn\overline{N}\,(r,f) + S(r,f),$$

where m is degree of highest order derivative occuring in it. Also by Lemma 4, we get

$$N\left(r,\frac{1}{P(f)}\right) \leq mn\left[\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right] + nN\left(r,\frac{1}{f}\right) + S(r,f).$$

Therefore,using lemma 4, (2) and (3) can be written as

$$T\,(r,f) \leq \overline{N}\left(r,\frac{1}{f}\right) + mn\left[\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right] + nN\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + S(r,f), \tag{4}$$

$$\begin{aligned} T(r,P(f)) \leq{}& mn\left[\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right] + nN\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f}\right) + nN(r,f) \\ &+ mn\overline{N}(r,f) + S(r,f). \end{aligned} \tag{5}$$

Adding (4) and (5) we get,

$$\begin{aligned} T\,(r,f) + T\,(r,P(f)) &\leq 2mn\left[\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)\right] + 2nN(r,\frac{1}{f}) + 2\overline{N}(r,\frac{1}{f}) \\ &\quad + nN(r,f) + (mn+1)\overline{N}(r,f) + S(r,f) \\ &\leq 2(mn+n+1)N\left(r,\frac{1}{f}\right) + (3mn+n+1)\,N(r,f) + S(r,f) \\ &\leq (3mn+2n+2)\left(N\left(r,\frac{1}{f}\right) + N(r,f)\right) + S(r,f) \\ &\leq S(r,f) \quad \text{which is contradiction, as the above relation implies} \end{aligned}$$

$$1 \leq \frac{S\,(r,f)}{T(r,f) + T(r,P(f))} \to 0, \quad as \quad r \to \infty,$$

possibly outside a set of finite measure. Hence, $k = 1$ Therefore

$$f \equiv P(f).$$

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