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A PERTURBATION TECHNIQUE TO COMPUTE INITIAL AMPLITUDE AND PHASE FOR THE KRYLOV-BOGOLIUBOV-MITROPOLSKII METHOD

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Abstract. Recently, a unified Krylov-Bogoliubov-Mitropolskii method has been presented (by Shamsul [1]) for solving an n-th, $n = 2$ or $n > 2$, order nonlinear differential equation. Instead of amplitude(s) and phase(s), a set of variables is used in [1] to obtain a general formula in which the nonlinear differential equations can be solved. By a simple variables transformation the usual form solutions (i.e., in terms of amplitude(s) and phase(s)) have been found. In this paper a perturbation technique is developed to calculate the initial values of the variables used in [1]. By the noted transformation the initial amplitude(s) and phase(s) can be calculated quickly. Usually the conditional equations are nonlinear algebraic or transcendental equations; so that a numerical method is used to solve them. Rink [7] earlier employed an asymptotic method for solving the conditional equations of a second-order differential equation; but his derived results were not so good. The new results agree with their exact values (or numerical results) nicely. The method can be applied whether the eigen-values of the unperturbed equation are purely imaginary, complex conjugate or real. Thus the derived solution is a general one and covers the three cases, i.e., un-damped, under-damped and over-damped.

1. Introduction

Recently, Shamsul [1] has presented a unified Krylov-Bogoliubov-Mitropolskii (KBM) method [2, 3, 4] for solving an n-th, $n = 2, 3, \ldots$, order differential equation with small nonlinearities. The method is a widely used tool to tackle nonlinear vibration problems. First, the method was presented (by Krylov and Bogoliubov [2]) for obtaining periodic solution of a second order differential equation. Then the method was amplified and justified by Bogoliubov and Mitropolskii [3]. Popov [5] extended the method to an under-damped case. Using the same method, Murty, Deekshatulu and Krisna [6] investigated the over-damped cases of the second- and fourth-order differential equations. Murty [4] also presented a unified KBM method for solving a second-order nonlinear differential equation.

Though the KBM method is a used tool in perturbation method, yet it has a major problem. The conditional equations (in which the initial amplitude(s) and phase(s) are calculated)

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appear in as nonlinear algebraic or transcendental equations. In general, these equations are solved by a numerical method. But Rink [7] earlier employed a perturbation tool to calculate the initial amplitude and phase of a second-order equation. He considered the over-damped case to illustrate his technique. He formulated his method for one of the Murty, Deekshatulu and Krisna's [5] solutions, which had covered a class of over-damped systems only. It is noted that Murty, Deekshtulu and Krisna found two over-damped solutions for two classes of over-damped systems (see [5] also [6] for details). However, shamsul's [1] unified solution covers all kinds of over-damped solutions as well as under-damped and un-damped. Though Rink [7] investigated a class of over-damped problems yet his results were not so good. He had determined the third approximation of the solution and carried out the calculations until third approximation and obtained a satisfactory result (theoretically, a third approximate solution is very close to exact or numerical solution). The aim of this paper is to develop a new perturbation technique, which gives more correct results of the initial amplitude(s) and phase(s). The method covers the un-damped, under-damped and over-damped cases. A single solution can be arbitrarily used for the three cases. Moreover, the method can be easily extended to an n-th, $n = 2,\ 3,\ \cdots$, order nonlinear differential equation. It should be noted that the Rink's technique is too difficult to calculate the amplitude(s) and phase(s) for a nonlinear differential equation possessing more than the second-derivative. On the contrary, the formulation as well as the determination of the solution (concern of this paper) is very simple.

2. Determination of solution and initial condition equations of a second-order equation

Let us consider the second-order autonomous equation

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}). \tag{2.1}$$

An asymptotic solution of this equation can be chosen in the form [1]

$$x(t,\ \varepsilon) = a_1(t)e^{\lambda_1 t} + a_2(t)\,e^{\lambda_2 t} + \varepsilon u_1(a_1, a_2) + \varepsilon^2 u_2(a_1, a_2) + \varepsilon^3 \cdots, \tag{2.2}$$

where $a_j,\quad j = 1,\ 2$ satisfy the first order differential equations

$$\dot{a}_j = \varepsilon A_j^{(1)}(a_1, a_2) + \varepsilon^2 A_j^{(2)}(a_1, a_2) + \varepsilon^3 A_j^{(3)}(a_1, a_2) + \varepsilon^4 \cdots \tag{2.3}$$

It is noted that $\lambda_j,\quad j = 1,\ 2$ are the eigen-values of the unperturbed equation of Eq. (2.1). Now Eq. (2.1) can be written as

$$(D - \lambda_1)(D - \lambda_2)x = \varepsilon f, D = d/dt. \tag{2.4}$$

Substituting solution Eq. (2.2) into the left side of Eq. (2.4), utilizing Eq. (2.3) and then equating the coefficients of ε^1, we obtain ([1], see also [8] for details)

$$(D - \lambda_2)(A_1^{(1)} e^{\lambda_1 t}) + (D - \lambda_1)(A_2^{(1)} e^{\lambda_2 t}) + (D - \lambda_1)(D - \lambda_2)u_1 = f^{(0)}(a_1, a_2, t). \tag{2.5}$$

It is clear that solution Eq. (2.2) is not considered in a usual form. This solution starts with two variables, $a_j(t)$, $j = 1, 2$, rather than amplitude and phase. Generally the used variables are respectively complex and real for the under-damped (or, un-damped) and over-damped cases. Shamsul [1] has used the set of variables a_j, $j = 1, 2, \cdots, n$ to present a general formula for solving an n-th order differential equation. Such choice of variables greatly facilitates the KBM method, since the formulation of the method is simple and the related equations to $\dot{a}_j$, $j = 1, 2$ and the functions $u_1, u_2, \cdots$ can be found in terms of a_j, $j = 1, 2$ quickly by imposing a restriction that $u_1, u_2, \cdots$ exclude the terms $a_1^{m_1} a_2^{m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$ where $m_1 \sim m_2 = 1$. This assumption guarantees that the terms $u_1, u_2, \cdots$ exclude the first harmonic terms as well as secular terms. By a simple variable transformation, namely, $a_1 = \frac{1}{2}\alpha e^{i\varphi}$, $a_2 = \frac{1}{2}\alpha e^{-i\varphi}$ or $a_1 = \frac{1}{2}\alpha e^{\varphi}$, $a_2 = \pm\frac{1}{2}\alpha e^{-\varphi}$($\alpha$ and φ are respectively amplitude and phase variables), all these equations and functions can be transformed to the usual forms (see [1, 8] for details).

2.1. Example

Let us consider the *Duffing* equation with a linear damping, $-2k\dot{x}$,

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = -\varepsilon x^3. \tag{2.6}$$

For Eq. (2.6), $f = -x^3$ and formula Eq. (2.5) becomes

$$\begin{aligned}(D-\lambda_2)(A_1^{(1)} e^{\lambda_1 t}) + (D-\lambda_1)(A_2^{(1)} e^{\lambda_2 t}) + (D-\lambda_1)(D-\lambda_2)u_1 \\ = -a_1^3 e^{3\lambda_1 t} - 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} - 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} - a_2^3 e^{3\lambda_2 t}.\end{aligned} \tag{2.7}$$

To eliminate the secular terms it has already been restricted that $u_1, u_2, \cdots$ do not contain the terms $a_1^{m_1} a_2^{m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}$ where $m_1 \sim m_2 = 1$. Therefore, Eq. (2.7) can be separated into three parts for $A_1^{(2.1)}$, $A_2^{(2.1)}$ and u_1 as

$$(D-\lambda_2)(A_1^{(1)} e^{\lambda_1 t}) = -3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t}, \tag{2.8}$$

$$(D-\lambda_1)(A_2^{(1)} e^{\lambda_2 t}) = -3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}, \tag{2.9}$$

$$(D-\lambda_1)(D-\lambda_2)u_1 = -a_1^3 e^{3\lambda_1 t} \_ a_2^3 e^{3\lambda_2 t}. \tag{2.10}$$

Solving the above three equations, we obtain

$$A_1^{(1)} = l_1 a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t}, A_2^{(1)} = l_1^* a_1 a_2^2 e^{(2\lambda_1+\lambda_2)t}, l_1 = -3/(2\lambda_1), l_1^* = -3/(2\lambda_2), \tag{2.11}$$

and

$$u_1 = c_1 a_1^3 e^{3\lambda_1 t} + c_1^* a_2^3 e^{3\lambda_2 t}, c_1 = -1/[2\lambda_1(3\lambda_1-\lambda_2)], c_1^* = -1/[2\lambda_2(3\lambda_2-\lambda_1)]. \tag{2.12}$$

Thus a first approximate solution of the Eq. (2.6) becomes

$$x(t,\varepsilon) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon(c_1 a_1^3 e^{3\lambda_1 t} + c_1^* a_2^3 e^{3\lambda_2 t}), \tag{2.13}$$

where a_1, a_2 satisfy

$$\dot{a}_1 = \varepsilon\, l_1 a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t}, \dot{a}_2 = \varepsilon\, l_1^* a_1 a_2^2 e^{(2\lambda_1+\lambda_2)t}, \tag{2.14}$$

and c_1, c_1^*; l_1, l_1^* are given by Eqs. (2.11)-(2.12). This solution is used arbitrarily for different values of λ_1, λ_2, whether they are real, complex conjugate or purely imaginary. Solution Eq. (2.13) represents the over-damped case when λ_1, λ_2 are real. In this case Eq. (2.13) can be written as the usual forms (presented by Murty et al. [4, 6]) by replacing the variables $a_1 = \frac{1}{2} a\, e^{\varphi}$, $a_2 = \pm\frac{1}{2} a\, e^{-\varphi}$ and substituting $\lambda_1 = -k+\omega$, $\lambda_2 = -k-\omega$, $\omega^2 = k^2-\omega_0^2$ (see [1]). On the contrary, the under-damped solution (early presented by Popov [5]) can be found from Eqs. (2.13) and (2.14) by replacing $a_1 = \frac{1}{2} a\, e^{i\varphi}$, $a_2 = \pm\frac{1}{2} a\, e^{-i\varphi}$and substituting $\lambda_1 = -k+i\omega$, $\lambda_2 = -k-i\omega$, $\omega^2 = \omega_0^2 - k^2$. If $k \to 0+$, the solution approaches to the original solution (un-damped) derived by KBM [2, 3]. It is noted that the over-damped solutions can be transformed to the under-damped and un-damped solutions replacing ω, φ, a by $i\omega$, $i\varphi$ or/and $-ia$. All these transformations are also possible for the initial conditions equations of the general solution Eq. (2.13).

The initial condition equations of Eq. (2.13) become

$$\begin{aligned} x_I &= x(0,\varepsilon) = a_{1,0} + a_{2,0} + \varepsilon(c_1 a_{1,0}^3 + c_1^* a_{2,0}^3),\\ \dot{x}_I &= \dot{x}(0,\varepsilon) = \lambda_1 a_{1,0} + \lambda_2 a_{2,0} + \varepsilon(3\lambda_1 c_1 a_{1,0}^3 + l_1 a_{1,0}^2 a_{2,0} + l_1^* a_{1,0} a_{2,0}^2 + 3\lambda_2 c_1^* a_{2,0}^3), \end{aligned} \tag{2.15}$$

where $a_1(0) = a_{1,0}$, $a_2(0) = a_{2,0}$. We can show that Eq. (2.15) is similar to that presented by Murty et al. [4, 6]. If we replace $a_{1,0} = \frac{1}{2} a_0\, e^{\varphi_0}$, $a_{2,0} = \pm\frac{1}{2} a_0\, e^{-\varphi_0}$, Eq. (2.15) becomes

$$\begin{aligned} x_I &= \frac{1}{2} a_0(e^{\varphi_0} \pm e^{-\varphi_0}) + \varepsilon\, a_0^3(c_1 e^{3\varphi_0} \pm c_1^* e^{-3\varphi_0})/8, \\ \dot{x}_I &= \frac{1}{2} a_0(\lambda_1 e^{\varphi_0} \pm \lambda_2 e^{-\varphi_0}) \pm \varepsilon\, a_0^3(l_1 e^{\varphi_0} \pm l_1^* e^{-\varphi_0})/8 + 3\varepsilon\, a_0^3(\lambda_1 c_1 e^{3\varphi_0} \pm \lambda_2 c_1^* e^{-3\varphi_0})/8. \end{aligned} \tag{2.16}$$

In an over- damped case, we can substitute $\lambda_1 = -k+\omega$, $\lambda_2 = -k-\omega$, in Eq. (2.16) and obtain

$$\begin{aligned} x_I &= a_0 \cosh\varphi_0 - \frac{\varepsilon\, a_0^3[(k^2+2\omega^2)\cosh 3\varphi_0 + 3k\omega \sinh 3\varphi_0]}{16\omega_0^2(k^2-4\omega^2)}, \\ \dot{x}_I &= a_0(-k\cosh\varphi_0 + \omega\sinh\varphi_0) + 3\varepsilon\, a_0^3\left(\frac{k\cosh\varphi_0 + \omega\sinh\varphi_0}{8\omega_0^2} + \frac{k\cosh 3\varphi_0 + 2\omega\sinh 3\varphi_0}{16(k^2-4\omega^2)}\right) \end{aligned} \tag{2.17}$$

or

$$\begin{aligned} x_I &= a_0 \sinh\varphi_0 - \frac{\varepsilon\, a_0^3[3k\omega\cosh 3\varphi_0 + (k^2+2\omega^2)\sinh 3\varphi_0]}{16\omega_0^2(k^2-4\omega^2)}, \\ \dot{x}_I &= a_0(\omega\cosh\varphi_0 - k\sinh\varphi_0) - 3\varepsilon\, a_0^3\left(\frac{\omega\cosh\varphi_0 + k\sinh\varphi_0}{8\omega_0^2} - \frac{2\omega\cosh 3\varphi_0 + k\sinh 3\varphi_0}{16(k^2-4\omega^2)}\right) \end{aligned} \tag{2.18}$$

Except notations, Eqs. (2.17) and (2.18) are identical to those obtained by Murty et al. [4, 6]. It is noted that $\varphi_0$is real for the over-damped cases while $\varphi_0$becomes purely imaginary for both under-damped and un-damped cases. For the given initial conditions $[x(0),\ \dot{x}(0)]$ or $[x_I,\ \dot{x}_I]$, one of two equations is solvable, either Eq. (2.17) or Eq. (2.18) (see [4, 6] for details). But both equations are solvable in under-damped and un-damped cases. By replacing ω, φ with $i\omega$, $i\varphi$, Eq. (2.17) becomes

$$\begin{aligned} x_I &= a_0\cos\varphi_0 - \frac{\varepsilon a_0^3[(k^2-2\omega^2)\cos 3\varphi_0 - 3k\omega\sin 3\varphi_0]}{16\omega_0^2(k^2+4\omega^2)}, \\ \dot{x}_I &= a_0(-k\cos\varphi_0+\omega\sin\varphi_0) + 3\varepsilon a_0^3\left(\frac{k\cos\varphi_0-\omega\sin\varphi_0}{8\omega_0^2} + \frac{k\cos 3\varphi_0 - 2\omega\sin 3\varphi_0}{16(k^2+4\omega^2)}\right). \end{aligned} \tag{2.19}$$

On the contary, Eq. (2.18) becomes

$$\begin{aligned} x_I &= a_0\sin\varphi_0 - \frac{\varepsilon a_0^3[3k\omega\cos 3\varphi_0 + (k^2-2\omega^2)\sin 3\varphi_0]}{16\omega_0^2(k^2+4\omega^2)}, \\ \dot{x}_I &= a_0(\omega\cos\varphi_0 - k\sin\varphi_0) + 3\varepsilon a_0^3\left(\frac{\omega\cos\varphi_0 + k\sin\varphi_0}{8\omega_0^2} - \frac{2\omega\cos 3\varphi_0 + k\sin 3\varphi_0}{16(k^2+4\omega^2)}\right). \end{aligned} \tag{2.20}$$

Herein $a$is also replaced by $-ia$. One can verify that both Eqs. (2.19) and (2.20) are solvable for every initial value problem. It is noted that $a_0$is same for both equations; but $\varphi_0$has different values. Thus one asymptotic solution covers all under-damped and un-damped cases, but two solutions are needed for over-damped case (see [4] for details).

3. Determination of asymptotic solution of initial condition equations

We can use two asymptotic series to solve Eq. (2.15) as

$$a_{j,0} = \tilde{a}_{j,0} + \varepsilon\alpha_j(\tilde{a}_{1,0},\tilde{a}_{2,0}) + \varepsilon^2\beta_j(\tilde{a}_{1,0},\tilde{a}_{2,0}) + \varepsilon^3\cdots, \quad j=1,2, \tag{3.1}$$

where $x_I = \tilde{a}_{1,0} + \tilde{a}_{2,0}$ and $\dot{x}_I = \lambda_1\tilde{a}_{1,0} + \lambda_2\tilde{a}_{2,0}$.

Substituting Eq. (3.1) into Eq. (2.15), simplifying and equating the coefficients of ε^1 and ε^2, we obtain

$$\begin{aligned} &\alpha_1+\alpha_2 = -(c_1\tilde{a}_{1,0}^3 + c_1^*\tilde{a}_{2,0}^3), \\ &\lambda_1\alpha_1+\lambda_2\alpha_2 = -3(\lambda_1 c_1\tilde{a}_{1,0}^3 + \lambda_2 c_1^*\tilde{a}_{2,0}^3) - \tilde{a}_{1,0}\tilde{a}_{2,0}(l_1\tilde{a}_{1,0} + l_1^*\tilde{a}_{2,0}), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &\beta_1+\beta_2 = -3(c_1\tilde{a}_{1,0}^2\alpha_1 + c_1^*\tilde{a}_{2,0}^2\alpha_2), \\ &\lambda_1\beta_1+\lambda_2\beta_2 = -9(\lambda_1 c_1\tilde{a}_{1,0}^2\alpha_1 + \lambda_2 c_1^*\tilde{a}_{2,0}^2\alpha_2) - \tilde{a}_{1,0}\tilde{a}_{2,0}(l_1\alpha_1 + l_1^*\alpha_2) \\ &\qquad -(\alpha_1\tilde{a}_{2,0} + \alpha_2\tilde{a}_{1,0})(l_1\tilde{a}_{1,0} + l_1^*\tilde{a}_{2,0}). \end{aligned} \tag{3.3}$$

The right hand sides of Eqs. (3.2)-(3.3) are real, since λ_1, λ_2 as well as c_1, c_1^*; l_1, l_1^* are occurred in conjugate pairs. For the over-damped case, $\alpha_1+\alpha_2$, $\lambda_1\alpha_1+\lambda_2\alpha_2,\cdots$,

$\lambda_1\beta_1+\lambda_2\beta_2$ can be calculated directly from Eqs. (3.2)-(3.3). However, in the case of under-damped or un-damped, these terms can be calculated easily replacing the right hand sides in real forms as

$$\alpha_1+\alpha_2=-2Re(c_1\tilde{a}_{1,0}^3), \lambda_1\alpha_1+\lambda_2\alpha_2=-6Re(\lambda_1 c_1\tilde{a}_{1,0}^3)-\tilde{a}_{1,0}\tilde{a}_{2,0}Re(l_1\tilde{a}_{1,0}), \tag{3.4}$$

and

$$\begin{aligned}&\beta_1+\beta_2=-6Re(c_1\tilde{a}_{1,0}^2\alpha_1)\,,\\&\lambda_1\beta_1+\beta_2\alpha_2=-18Re(\lambda_1 c_1\tilde{a}_{1,0}^2\alpha_1)-2\,\tilde{a}_{1,0}\tilde{a}_{2,0}\,Re(l_1\alpha_1)-4Re(\alpha_1\tilde{a}_{2,0})\times Re(l_1\tilde{a}_{1,0})\,.\end{aligned} \tag{3.5}$$

Solving Eqs. (3.2)-(3.3) or Eqs. (3.4)-(3.5), we obtain $\alpha_1,\ \alpha_2;\ \beta_1,\ \beta_2$ which complete the second approximation of the initial condition equations Eq. (2.15).

4. Rink's [7] solution

To solve the transformed initial condition equations Eq. (2.17), Rink [7] had chosen the following asymptotic series

$$\begin{aligned}a_I&=\tilde{a}_0+\mu\,\tilde{a}_1(\tilde{a}_0,\tilde{\psi}_0)+\varepsilon^2\tilde{a}_2(\tilde{a}_0,\tilde{\psi}_0)+\varepsilon^3\cdots,\\ \psi_I&=\tilde{\psi}_0+\mu\,\psi_1(\tilde{a}_0,\tilde{\psi}_0)+\varepsilon^2\psi_2(\tilde{a}_0,\tilde{\psi}_0)+\varepsilon^3\cdots,\end{aligned} \tag{4.1}$$

where $\tilde{a}_0,\ \tilde{\psi}_0$ satisfy the equations $x_I=\tilde{a}_0\cosh\tilde{\psi}_0$ and $\dot{x}_I=-k\tilde{a}_0\cosh\tilde{\psi}_0+\omega\,\tilde{a}_0\sinh\tilde{\psi}_0$, or $\tilde{a}_0=[x_I^2-((\dot{x}_I+kx_I)/\omega)^2]^{\frac{1}{2}},\ \tilde{\psi}_0=\tanh^{-1}((\dot{x}_I+kx_I)/\omega)$.

First, Rink had determined a third approximate solution of equation, $\ddot{x}+3\dot{x}+2x=\mu x^3$, $\mu<<1$ and then substituting the series Eq. (4.1) into the initial conditions equations of that solution, he calculated $a_1,\ \psi_1$ to third approximation to obtain a satisfactory result. Definitely that was a difficult task. But we are not interested to determine higher approximate solution. The first approximation solution is usually applied in a practical problem when the damping force is significant, since the derivation of the formula as well as determination of the solution of a nonlinear differential equation is difficult (see [4, 9] for details). We only calculate $a_I,\ \psi_I$ to second approximation from Eqs. (3.2)-(3.3) or (3.4)-(3.5) for the first approximation solution Eq. (2.13). All these results show a good agreement with the numerical results. Rink's solution (truncated form for the first approximation) is

$$x(t,\ \mu)=a\cosh\psi+\mu\,a^3(11\cosh 3\psi-9\sinh 3\psi)/160, \tag{4.2}$$

where a and ψ satisfy the differential equations

$$\dot{a}=-3a/2-9\mu\,a^3/32, \dot{\psi}=-1/2+3\mu\,a^2/32. \tag{4.3}$$

For this solution, the initial conditional equations become

$$\begin{aligned} x_I &= a_0 \cosh\psi_0 + \mu\, a_0^3 (11\cosh 3\psi_0 - 9\sinh 3\psi_0)/160\,, \\ \dot{x}_I &= -a_0(3\cosh\psi_0 + \sinh\psi_0)/2 \\ &\quad -3\mu\, a_0^3 (15\cosh\psi_0 - 5\sinh\psi_0 - 12\cosh 3\psi_0 + 8\sinh 3\psi_0)/160 \end{aligned} \tag{4.4}$$

Solution Eq. (4.2) as well as initial conditions Eq. (4.4) was determined by Murty, Deekatulu and Krisna [6]. Rink [7] formulated and determined the following results of a_I, ψ_I for Eq. (4.4) (only first order solution is given)

$$\begin{aligned} a_I &= \tilde{a}_0 + \mu\, \tilde{a}_0^3(-10\cosh 2\tilde{\psi}_0 - 16\cosh 4\tilde{\psi}_0 + 30\sinh 2\tilde{\psi}_0 + 24\sinh 4\tilde{\psi}_0 + 15)/160, \\ \psi_I &= \tilde{\psi}_0 + \mu\, \tilde{a}_0^2(-60\cosh 2\tilde{\psi}_0 - 24\cosh 4\tilde{\psi}_0 + 20\sinh 2\tilde{\psi}_0 + 16\sinh 4\tilde{\psi}_0 - 45)/160\,. \end{aligned} \tag{4.5}$$

4.1. Determination of Rink's [7] solution from our solution

For the equation, $\ddot{x} + 3\dot{x} + 2x = \mu x^3$, $\mu << 1$ [4, 6, 7], $\varepsilon = -\mu$, $\lambda_1 = -2$, $\lambda_2 = -1$. Therefore, we can choose $k = 3/2$ and $\omega = -1/2$. Substituting these values of k, ω into Eq. (2.17), we obtain

$$\begin{aligned} x_I &= a_0 \cosh\varphi_0 - \varepsilon\, a_0^3 (11\cosh 3\varphi_0 - 9\sinh 3\varphi_0)/160, \\ \dot{x}_I &= -a_0(3\cosh\varphi_0 + \sinh\varphi_0)/2 \\ &\quad -3\varepsilon\, a_0^3 (15\cosh\varphi_0 - 5\sinh\varphi_0 - 12\cosh 3\varphi_0 + 8\sinh 3\varphi_0)/160\,. \end{aligned} \tag{4.6}$$

It is noted that the used variables of our solution are not same to Murty et al.'s [4, 6]. But it has been proved that Eq.(2.13) and Eq. (4.2) are identical (see [1]). However, the initial conditions equations Eq. (4.4) and Eq. (4.6) are same, if we only replace $\varphi_0$by $\psi_0$and ε by $-\mu$.

Now we shall find Rink's solution of Eq.(4.5) from our solution. Substituting $\lambda_1 = -2$ and $\lambda_2 = -1$, Eqs. (3.2)-(3.3) become

$$\begin{aligned} \alpha_1 + \alpha_2 &= \tilde{a}_{1,0}^3/20 + \tilde{a}_{2,0}^3/2 \\ 2\alpha_1 + \alpha_2 &= 3(\tilde{a}_{1,0}^3/10 + \tilde{a}_{2,0}^3/2) + 3\,\tilde{a}_{1,0}\tilde{a}_{2,0}(3\,\tilde{a}_{1,0}/4 + \tilde{a}_{2,0}/2). \end{aligned} \tag{4.7}$$

The solution of Eq. (4.7) is

$$\begin{aligned} \alpha_1 &= \tilde{a}_{1,0}^3/4 + 3\,\tilde{a}_{1,0}^3\tilde{a}_{2,0}/4 + 3\,\tilde{a}_{1,0}\tilde{a}_{2,0}^2/2 + \tilde{a}_{2,0}^3\,, \\ \alpha_2 &= -(\tilde{a}_{1,0}^3/5 + 3\,\tilde{a}_{1,0}^2\, a_{2,0}/4 + 3\, a_{1,0}\tilde{a}_{2,0}^2/2 + \tilde{a}_{2,0}^3/2). \end{aligned} \tag{4.8}$$

Substituting the values of α_1, α_2 from Eq. (4.8) into Eq. (2.19), the first order solution of the conditional equations Eq. (2.15) can be found as

$$\begin{aligned} a_{1,I} &= \tilde{a}_{1,0} + \varepsilon\,(\tilde{a}_{1,0}^3/4 + 3\,\tilde{a}_{1,0}^2\tilde{a}_{2,0}/4 + 3\tilde{a}_{1,0}\tilde{a}_{2,0}^2/2 + \tilde{a}_{2,0}^3), \\ a_{2,I} &= \tilde{a}_{2,0} - \varepsilon\,(\tilde{a}_{1,0}^3/5 + 3\,\tilde{a}_{1,0}^2\tilde{a}_{2,0}/4 + 3\tilde{a}_{1,0}\tilde{a}_{2,0}^2/2 + \tilde{a}_{2,0}^3/2). \end{aligned} \tag{4.9}$$

Replacing $a_{1,0}=\frac{1}{2}a_I e^{\varphi_I},\ a_{2,0}=\frac{1}{2}a_I e^{-\varphi_I}$ and $\tilde{a}_{1,0}=\frac{1}{2}\tilde{a}_0 e^{\tilde{\varphi}_0}, \tilde{a}_{2,0}=\frac{1}{2}\tilde{a}_0 e^{-\tilde{\varphi}_0}$ and simplifylng, we obtain

$$\begin{aligned} a_I e^{\varphi_I} &= \tilde{a}_0 e^{\tilde{\varphi}_0}+\varepsilon\,\tilde{a}_0^3(e^{3\tilde{\varphi}_0}/4+3e^{\tilde{\varphi}_0}/4+3e^{-\tilde{\varphi}_0}/2+e^{-3\tilde{\varphi}_0})/4,\\ a_I e^{-\varphi_I} &= \tilde{a}_0 e^{-\varphi_0}-\varepsilon\,\tilde{a}_0^3(e^{3\tilde{\varphi}_0}/5+3e^{\tilde{\varphi}_0}/4+3e^{-\tilde{\varphi}_0}/2+e^{-3\tilde{\varphi}_0}/2)/4. \end{aligned} \tag{4.10}$$

Now substituting $a_I=\tilde{a}_0+\varepsilon\,p+\varepsilon^2\cdots$, $\varphi_I=\tilde{\varphi}_0+\varepsilon\,q+\varepsilon^2\cdots$ and expanding $e^{\pm\varepsilon\,q}$ in *Maclaurin's series,* we obtain

$$\begin{aligned} (\tilde{a}_0+\varepsilon\,p+\cdots)(1+\varepsilon\,q+\cdots)e^{\tilde{\varphi}_0} &= \tilde{a}_0 e^{\tilde{\varphi}_0}+\varepsilon\,a_0^3(e^{3\tilde{\varphi}_0}/4+3e^{\tilde{\varphi}_0}/4+3e^{-\tilde{\varphi}_0}/2+e^{-3\tilde{\varphi}_0})/4,\\ (\tilde{a}_0+\varepsilon\,p+\cdots)(1-\varepsilon\,q+\cdots)e^{-\tilde{\varphi}_0} &= \tilde{a}_0 e^{-\tilde{\varphi}_0}-\varepsilon\,a_0^3(e^{3\tilde{\varphi}_0}/5+3e^{\tilde{\varphi}_0}/4+3e^{-\tilde{\varphi}_0}/2+e^{-3\tilde{\varphi}_0}/2)/4. \end{aligned} \tag{4.11}$$

Equation the coefficients of ε on both sides of Eq. (4.11), we obtain

$$\begin{aligned} p+\tilde{a}_0 q &= \tilde{a}_0^3(e^{2\tilde{\varphi}_0}/4+3/4+3e^{-2\tilde{\varphi}_0}/2+e^{-4\varphi_0})/4,\\ p-\tilde{a}_0 q &= -\tilde{a}_0^3(e^{4\tilde{\varphi}_0}/5+3e^{2\tilde{\varphi}_0}/4+3/2+e^{-2\tilde{\varphi}_0}/2)/4, \end{aligned} \tag{4.12}$$

or,

$$\begin{aligned} p &= -\tilde{a}_0^3(10e^{2\tilde{\varphi}_0}-20e^{-2\tilde{\varphi}_0}+4e^{4\tilde{\varphi}_0}-20e^{-4\tilde{\varphi}_0}+15)/160,\\ q &= -\tilde{a}_0^2(20e^{2\tilde{\varphi}_0}+40e^{-2\tilde{\varphi}_0}+20e^{4\tilde{\varphi}_0}+40e^{-4\tilde{\varphi}_0}+45)/160. \end{aligned} \tag{4.13}$$

or,

$$\begin{aligned} p &= -\tilde{a}_0^3(-10\cosh 2\tilde{\varphi}_0-16\cosh 4\tilde{\varphi}_0+30\sinh 2\tilde{\varphi}_0+24\sinh 4\tilde{\varphi}_0+15)/160,\\ q &= -\tilde{a}_0^2(-60\cosh 2\tilde{\varphi}_0-24\cosh 4\tilde{\varphi}_0+20\sinh 2\tilde{\varphi}_0+16\sinh 4\tilde{\varphi}_0-45)/160. \end{aligned} \tag{4.14}$$

Substituting these values, we obtain

$$\begin{aligned} a_I &= a_0-\varepsilon\,a_0^3(-10\cosh 2\varphi_0-16\cosh 4\varphi_0+30\sinh 2\varphi_0+24\sinh 4\varphi_0+15)/160,\\ \varphi_I &= \varphi_0-\varepsilon\,a_0^2(-60\cosh 2\varphi_0-24\cosh 4\varphi_0+20\sinh 2\varphi_0+16\sinh 4\varphi_0-45)/160. \end{aligned} \tag{4.15}$$

Except notations, Eqs. (4.5) and (4.15) are identical. Thus the first order Rink's solution has been determined from our solution. In a similar way we can determine Rink's higher order solution; but each step it needs the truncation of $e^{\pm\varepsilon\,q}$.

5. A third order nonlinear problem

Let us consider the nonlinear mechanical system governed by

$$\begin{aligned} m\ddot{x}+\sigma &= 0,\\ \sigma+\gamma\dot{\sigma} &= gx+h\dot{x}+ex^3\,, \end{aligned} \tag{5.1}$$

where x is the deformation, m is the mass of the system, and γ, g, $h$and $e$are positive constants. The terms with coefficients g and e(small) represent respectively the linear and nonlinear elasticity, the term with coefficient h corresponds to the linear viscous damping, and the term with coefficient γ reflects the linear relaxation.

Herein x satisfies a third-order nonlinear differential equation [eliminating σ from two equations of (5.1)]

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = -\varepsilon x^3. \tag{5.2}$$

where $k_1 = \gamma^{-1}$, $k_2 = h\gamma^{-1}m^{-1}$, $k_3 = g\gamma^{-1}m^{-1}$ and $\varepsilon = e\gamma^{-1}$. For this equation the first approximate solution is

$$x(t,\varepsilon) = \tilde{a}_1 e^{\lambda_1 t} + \tilde{a}_2 e^{\lambda_2 t} + \tilde{a}_3 e^{\lambda_3 t} + \varepsilon\,(h_2\tilde{a}_1\tilde{a}_2^2 e^{(\lambda_1+2\lambda_2)t} + h_2^*\tilde{a}_1\tilde{a}_3^2 e^{(\lambda_1+2\lambda_3)t} + h_3\tilde{a}_2^3 e^{3\lambda_1 t} + h_3^*\tilde{a}_3^3 e^{3\lambda_2 t}) \tag{5.3}$$

where λ_1, λ_2, λ_3 are eigen-values of unperturbed equation of Eq. (5.2) and a_1, a_2, $a_3$satisfy

$$\begin{aligned} \dot{a}_1 &= \varepsilon(l_2 a_1^3 + l_3 a_1 a_2 a_3), \\ \dot{a}_2 &= \varepsilon(m_1 a_1^2 a_2 + m_2 a_2^2 a_3), \\ \dot{a}_3 &= \varepsilon(m_1^* a_1^2 a_2 + m_2^* a_2^2 a_3). \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} l_2 &= [(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)]^{-1}, & l_3 &= 6[(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)]^{-1}, \\ m_1 &= 3[(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)]^{-1}, & m_1^* &= 3[(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)]^{-1}, \\ m_2 &= 3[2\lambda_2(2\lambda_2+\lambda_3-\lambda_1)]^{-1}, & m_2^* &= 3[2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)]^{-1} \\ h_2 &= 3[2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)]^{-1}, & h_2^* &= 3[2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)]^{-1}, \\ h_3 &= [2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)]^{-1}, & h_3^* &= [2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)]^{-1}. \end{aligned} \tag{5.5}$$

The initial condition equations of solution Eq. (5.3) are

$$\begin{aligned} x_I &= \tilde{a}_{1,0} + \tilde{a}_{2,0} + \tilde{a}_{3,0} + \varepsilon(h_2\tilde{a}_{1,0}\tilde{a}_{2,0}^2 + h_2^*\tilde{a}_{1,0}\tilde{a}_{3,0}^2 + h_3\tilde{a}_{2,0}^3 + h_3^*\tilde{a}_{3,0}^3), \\ \dot{x}_I &= \lambda_1\tilde{a}_{1,0} + \lambda_2\tilde{a}_{2,0} + \lambda_3\tilde{a}_{3,0} + \varepsilon(l_2\tilde{a}_{1,0}^3 + l_3\tilde{a}_{1,0}\tilde{a}_{2,0}\tilde{a}_{3,0} + m_1\tilde{a}_{1,0}^2\tilde{a}_{2,0} + m_2\tilde{a}_{2,0}^2\tilde{a}_{3,0} + m_1^*\tilde{a}_{1,0}^2\tilde{a}_{3,0} \\ &\quad + m_2^*\tilde{a}_2\tilde{a}_3^2 + (\lambda_1+2\lambda_2)h_2\tilde{a}_{1,0}\tilde{a}_{2,0}^2 + (\lambda_1+2\lambda_3)h_2^*\tilde{a}_{1,0}\tilde{a}_{3,0}^2 + 3\lambda_2 h_3\tilde{a}_{2,0}^3 + 3\lambda_2 h_3^*\tilde{a}_{3,0}^3) \\ \ddot{x}_I &= \lambda_1^2\tilde{a}_{1,0} + \lambda_2^2\tilde{a}_{2,0} + \lambda_3^2\tilde{a}_{3,0} + \varepsilon[4\lambda_1 l_2\tilde{a}_{1,0}^3 + (2\lambda_1+\lambda_2+\lambda_3)l_3\tilde{a}_{1,0}\tilde{a}_{2,0}\tilde{a}_{3,0} \\ &\quad + 2(\lambda_1+\lambda_2)m_1\tilde{a}_{1,0}^2\tilde{a}_{2,0} + (3\lambda_2+\lambda_3)m_2\tilde{a}_{2,0}^2\tilde{a}_3 + 2(\lambda_1+\lambda_3)m_1^*\tilde{a}_{1,0}^2\tilde{a}_{3,0} + (\lambda_2+3\lambda_3)m_2^*\tilde{a}_2\tilde{a}_{3,0}^2 \\ &\quad + (\lambda_1+2\lambda_2)^2 h_2\tilde{a}_{1,0}\tilde{a}_{2,0}^2 + (\lambda_1+2\lambda_3)^2 h_2^*\tilde{a}_{1,0}\tilde{a}_2^3 + 9\lambda_2^2 h_3\tilde{a}_{2,0}^3 + 9\lambda_3^2 h_3^*\tilde{a}_{3,0}^3] \end{aligned} \tag{5.6}$$

Substituting Eq. $a_{j,0} = \tilde{a}_{j,0} + \varepsilon\alpha_j(\tilde{a}_{1,0},\tilde{a}_{2,0}) + \varepsilon^2\beta_j(\tilde{a}_{1,0},\tilde{a}_{2,0}) + \varepsilon^3\cdots$, $j = 1, 2, 3$, into Eq. (5.6), simplifying and equating the coefficients of ε^1 and ε^2, we obtain

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3 &= -[h_2\tilde{a}_{1,0}\tilde{a}_{2,0}^2 + h_2^*\tilde{a}_{1,0}\tilde{a}_{3,0}^2 + h_3\tilde{a}_{2,0}^3 + h_3^*\tilde{a}_{3,0}^3], \\ \lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3 &= -[l_2\tilde{a}_{1,0}^3 + l_3\tilde{a}_{1,0}\tilde{a}_{2,0}\tilde{a}_{3,0} + \tilde{a}_{1,0}^2(m_1\tilde{a}_{2,0} + m_1^*\tilde{a}_{3,0}) \\ &\quad + \tilde{a}_{2,0}\tilde{a}_{3,0}(m_2\tilde{a}_{2,0} + m_2^*\tilde{a}_{3,0}) + a_{1,0}\{(\lambda_1+2\lambda_2)h_2\tilde{a}_{2,0}^2 \\ &\quad + (\lambda_1+2\lambda_3)h_2^*\tilde{a}_{3,0}^2\} + 3(\lambda_2 h_3\tilde{a}_{2,0}^3 + \lambda_3 h_3^*\tilde{a}_{3,0}^3)], \end{aligned} \tag{5.7}$$

$$\begin{aligned}\lambda_1^2\alpha_1+\lambda_2^2\alpha_2+\lambda_3^2\alpha_3 = &-[4\lambda_1 l_2\tilde{a}_{1,0}^3+(2\lambda_1+\lambda_2+\lambda_3)l_3\tilde{a}_{1,0}\tilde{a}_{2,0}\tilde{a}_{3,0}+2\tilde{a}_{1,0}^2\{(\lambda_1+\lambda_2)m_1\tilde{a}_{2,0}\\ &+(\lambda_1+\lambda_3)m_2^*\tilde{a}_{3,0}\}+\tilde{a}_{2,0}\tilde{a}_{3,0}\{(3\lambda_2+\lambda_3)m_2\tilde{a}_{2,0}+(\lambda_2+3\lambda_3)m_2^*\tilde{a}_{3,0}\}\\ &+a_{1,0}\{(\lambda_1+2\lambda_2)^2h_2\tilde{a}_{2,0}^2+(\lambda_1+2\lambda_3)^2h_2^*\tilde{a}_{3,0}^2\}+9(\lambda_2^2h_3\tilde{a}_{2,0}^3+\lambda_3^2h_3^*\tilde{a}_{3,0}^3)],\end{aligned}$$

and

$$\begin{aligned}\beta_1+\beta_2+\beta_3 = &-[2\tilde{a}_{1,0}(h_2\tilde{a}_{2,0}\alpha_2+h_2^*\tilde{a}_{3,0}\alpha_3)+\alpha_I(h_2\tilde{a}_{2,0}^2+h_2^*\tilde{a}_{3,0}^2)+3(h_3\tilde{a}_{2,0}^2\alpha_2+h_3^*\tilde{a}_{3,0}^2\alpha_3),\\ \lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3 = &-[3l_2\tilde{a}_{1,0}^2\alpha_1+l_3(\tilde{a}_{1,0}\tilde{a}_{2,0}\alpha_3+\tilde{a}_{1,0}\tilde{a}_{3,0}\alpha_2+\tilde{a}_{2,0}\tilde{a}_{3,0}\alpha_1)\\ &+\tilde{a}_{1,0}\{(\tilde{a}_{1,0}\alpha_2+2\tilde{a}_{2,0}\alpha_1)m_1+(\tilde{a}_{1,0}\alpha_3+2\tilde{a}_{3,0}\alpha_1)m_1^*\}\\ &+2\tilde{a}_{2,0}\tilde{a}_{3,0}(m_2\alpha_2+m_2^*\alpha_3)+m_2\tilde{a}_{2,0}^2\alpha_3+m_2^*\tilde{a}_{3,0}^2\alpha_2\\ &+\{(\lambda_1+2\lambda_2)(2\tilde{a}_{1,0}\alpha_2+\tilde{a}_{2,0}\alpha_1)h_2\tilde{a}_{2,0}\\ &+(\lambda_1+2\lambda_3)(\tilde{a}_{1,0}\alpha_3+\tilde{a}_{3,0}\alpha_1)h_2^*\tilde{a}_{3,0}+9(\lambda_2h_3\tilde{a}_{2,0}^2\alpha_2+\lambda_3h_3^*\tilde{a}_{3,0}^2\alpha_3), \quad (5.8)\\ \lambda_1^2\beta_1+\lambda_2^2\beta_2+\lambda_3^2\beta_3 = &-[12l_1\lambda_1\tilde{a}_{1,0}^2\alpha_1+l_3(2\lambda_1+\lambda_2+\lambda_3)(\tilde{a}_{1,0}\tilde{a}_{2,0}\alpha_3+\tilde{a}_{1,0}\tilde{a}_{3,0}\alpha_2+\tilde{a}_{2,0}\tilde{a}_{3,0}\alpha_1)\\ &2\tilde{a}_{1,0}\{m_1(\lambda_1+\lambda_2)(\tilde{a}_{1,0}\alpha_2+2\tilde{a}_{2,0}\alpha_1)+m_1^*(\lambda_1+\lambda_3)(\tilde{a}_{1,0}\alpha_3+2\tilde{a}_{3,0}\alpha_1)\\ &+2\tilde{a}_{2,0}\tilde{a}_{3,0}\{(3\lambda_2+\lambda_3)m_2\alpha_2+(\lambda_2+3\lambda_3)m_2^*\alpha_3\}\\ &+(3\lambda_2+\lambda_3)m_2\tilde{a}_{2,0}^2\alpha_3+(\lambda_2+3\lambda_3)m_2^*\tilde{a}_{3,0}^2\alpha_2\}\\ &\{(\lambda_1+2\lambda_2)^2(2\tilde{a}_{1,0}\alpha_2+\tilde{a}_{2,0}\alpha_1)h_2\tilde{a}_{2,0}\\ &+(\lambda_1+2\lambda_3)^2(2\tilde{a}_{1,0}\alpha_3+\tilde{a}_{3,0}\alpha_1)h_2^*\tilde{a}_{3,0}\}\\ &+27(\lambda_2^2h_3\tilde{a}_{2,0}^2\alpha_2+\lambda_3^2h_3^*\tilde{a}_{2,0}^2\alpha_3)].\end{aligned}$$

The right hand sides of Eqs. (5.7)-(5.8) are real as of Eqs. (3.2)-(3.3). Solving Eqs. (5.7)-(5.8), we obtain α_1, α_2, α_3; β_1, β_2, β_3 which complete the second approximation of the initial condition equations Eq. (5.6). We can transform these equations to real form; but it is no difficult to calculate these terms (without transformation) whether λ_1, λ_2, $\lambda_3$are real, complex or purely imaginary.

6. Results and discussion

In general Eqs. (2.17) or (2.18) or (2.19) or (2.20) is solved by a numerical method. Rink [7] employed a perturbation technique to solved Eq.(2.18); but his solution converges very slowly to the real solution. So, it is difficult to apply his [7] method in particular problems. In this paper, we have provided a new perturbation technique to solve the same equations. The new solution converges faster than that of Rink. It has been shown that Rink's solution is a truncated version of our solution (see **sub-section 4.1**). The main advantage of this method is that it can be quickly extended in third-, fourth-, etc. order problems. For the initial$\ddot{x}+3\dot{x}+2x=\mu x^3$, $\mu=0.1$, $[x(0)=2.2888, \quad \dot{x}(0)=-4.2865]$. Rink [7] found that the initial values of amplitude and phase are $a_I=2.12, \quad \psi_I=0.415$ (first approximation); but the exact value are $a_0 = 2.0$, $\psi_0 = 0.5$ (which completely satisfy initial condition equations Eq. (27)). On the contrary, we have found from Eq. (4.9) that $a_I = 1.9493$ and $\varphi_I = 0.5465$ ($= \psi_I$)(see **Appendix A**). These results are better than those second approximate values derived by Rink (which are $a_I = 1.96$, $\psi_I = 0.578$, see [7]). But using our technique we have found the second approximation results as $a_I = 1.9955$ and $\varphi_I = 0.5122$, which are much better results than the third approximate results obtained by Rink (Note: Rink's third approximate results are $a_I = 1.96$, $\psi_I = 0.578$).

In an under-damped case the results also rapidly converge to their exact values. For the initial value problem $\ddot{x} + \dot{x} + x = -\varepsilon x^3$, $\varepsilon = 0.1$, $[x(0) = 1.0,\ \dot{x}(0) = 0]$, we have calculated (see **Appendix B**) the initial amplitude and phase as $a_I = 1.123296$, $\varphi_I = -0.465357$ (first approximation) and the numerical results (obtain from Eq. (2.19) by *Newton-Raphson* formula) are $a_0 = 1.125335$, $\varphi_0 = -0.470666$. We have also calculated second approximate results as $a_I = 1.125595$, $\varphi_I = -0.471295$. It is interesting to note that the results converge more rapidly when the system is un-damped. For the initial value problem $\ddot{x} + x = -\varepsilon x^3$, $\varepsilon = 0.1$, $[x(0) = 1.0,\ \dot{x}(0) = 0]$, we have calculated the initial amplitude and phase as $a_I = 0.996875$, $\varphi_I = 0$ (first approximation) and the numerical results (obtained from Eq. (2.19) with $k = 0$, by *Newton-Raphson* formula) are $a_0 = 0.996904$, $\varphi_0 = 0$. Then we calculated second approximation as $a_I = 0.996904$, $\varphi_I = 0$, which are similar to the numerical solution up to six decimal places.

For the initial value problem (considered from [1]), $\dddot{x} + 0.9\ddot{x} + 1.24\dot{x} + 0.52x = -\varepsilon x^3$, $\varepsilon = 0.1$, $[x(0) = 1.49125,\ \dot{x}(0) = -0.43631,\ \ddot{x}(0) = -0.72346]$, we have calculated the initial amplitudes and phase as $a_I = 0.4817$, $b_I = 1.0184$, $\varphi_I = 0.0066$ (first approximation) utilizing formulae Eq. (5.7). The exact results are $a_0 = 0.5$, $b_0 = 1.0$, $\varphi_0 = 0$. In this case the first approximate results of b_I, φ_I are close to b_0, φ_0; but a_I is far from a_0. So it needs second approximation. We calculated these as $a_I = 0.50033$, $b_I = 1.00009$, $\varphi_I = 0.00003$. Which are almost equal to the exact values. However, we can show that the first approximate results are very close to the numerical results when the modulus of the real eigen-value is an order of 1. Let us consider another third-order initial value problem (from [9]). $\dddot{x} + \ddot{x} + 4\dot{x} + 4x = \mu x^3$, $\mu = 0.1$, $[x(0) = 1.,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -2]$. For this problem, we have computed $a_I = 0.40486$, $b_I = 0.62802$, $\varphi_I = 0.32723$. Their exact results are $a_0 = 0.40449$, $b_0 = 0.62845$, $\varphi_0 = 0.32608$. It is clear that the first order approximate results of both amplitudes and phase are very close to the numerical results.

Seeing all these problems we can decide that the second approximate results (sometimes first approximation) of the amplitude(s) and phase are very close to the numerical solutions whether the nonlinear system possesses the second or third derivative. The method can be easily applied to fourth- or more than fourth-order nonlinear differential systems. Thus it is no need to calculate the amplitude(s) and phase(s) by a numerical technique.

Appendix A

The eigen-values of $\ddot{x}+3\dot{x}+2x=0$ are $\lambda_1=-2,\ \lambda_2=-1$.Therefore, for initial conditions $[x[0]=2.2888,\ \ \dot{x}(0)=-4.2865]$, we have $\tilde{a}_{1,0}+\tilde{a}_{2,0}=2.2888,\ -\tilde{a}_{1,0}-2\tilde{a}_{2,0}=-4.28650$, or, $\tilde{a}_{1,0}=1.9977,\ \ \tilde{a}_{2,0}=0.2911$. For the nonlinear eqation $\ddot{x}+3\dot{x}+2x=-\varepsilon x^3$, we have calculated the following results:

$$c_1=-1/20,\quad c_1^*=-1/2,\quad l_1=3/4,\quad l_1^*=3/2.$$

Substituting these values into Eq. (3.2) and simplifying, we obtain

$$\alpha_1+\alpha_2=0.41096,\quad -\alpha_1-2\alpha_2=-3.55395. \tag{A.1}$$

The solution of Eq. (A.1) is

$$\alpha_1=3.1430,\quad \alpha_2=-2.7320 \tag{A.2}$$

Substituting these values of $\tilde{a}_{1,0},\ \ \tilde{a}_{2,0},\ \ \alpha_1,\ \ \alpha_2$together with $\varepsilon=-0.1$(since Murty, Deekshatulu and Krisna's [6] paper it was given that $\mu=0.1$ and $\mu=-\varepsilon$) and simplifying, we obtain $a_{1,0}=1.6834,\ \ a_{2,0}=0.5643$. From the approximate result of $a_{1,0}$ and $a_{2,0}$, we can easily calculate initial amplitude and phase as $a_I=1.9493,\ \ \varphi_I=0.5465$ since $a_{1,0}=a_I e^{\varphi_I}$ and $a_{2,0}=a_I e^{-\varphi_I}$.

Appendix B

The eigen- values of $\ddot{x}+\dot{x}+x=0$are $\lambda_1=(-1+i\sqrt{3})/2,\ \ \lambda_2=(-1-i\sqrt{3})/2$. Therefore, for initial conditions $[x(0)=1.0,\ \ \dot{x}(0)=0]$.
We have $\tilde{a}_{1,0}+\tilde{a}_{2,0}=1$, $(-1+i\sqrt{3})\tilde{a}_{1,0}+(-1-i\sqrt{3})\tilde{a}_{2,0}=0$, or, $\tilde{a}_{1,0}=(\sqrt{3}-i)/2\sqrt{3}$, $\tilde{a}_{2,0}=(\sqrt{3}+i)/2\sqrt{3}$. For the nonlinear equation $\ddot{x}+\dot{x}+x=-\varepsilon x^3$, we have calculated the following results:

$$c_1=(5-i3\sqrt{3})/52,\quad c_1^*=(5+i3\sqrt{3})/52,\quad l_1=3(1+i\sqrt{3})/4,\quad l_1^*=3(1-i\sqrt{3})/4.$$

Substituting these values into Eq. (3.2) and simplifying, we obtain

$$\alpha_1+\alpha_2=1/26\,,\quad (-1+i\sqrt{3})\alpha_1+(-1-i\sqrt{3})\alpha_2=-17/13 \tag{B.1}$$

The solution of Eq. (B.1) is

$$\alpha_1=(\sqrt{3}+33i)/52\sqrt{3},\quad \alpha_2=(\sqrt{3}-33i)/52\sqrt{3} \tag{B.2}$$

Substituting these values of $\tilde{a}_{1,0}$, $\tilde{a}_{2,0}$, α_1, α_2 together with $\varepsilon = 0.1$ and simplifying, we obtain $a_{1,0} = 0.501923 - 0.252035i$, $a_{2,0} = 0.501923 + 0.252035i$. From the approximate result of $a_{1,0}$ or $a_{2,0}$, we can easily calculate initial amplitude and phase as $a_I = 1.123296$, $\varphi_I = -0.465357$, since $a_{1,0} = a_I e^{i\varphi_I}$ and $a_{2,0} = a_I e^{-i\varphi_I}$.

In a similar way, we can determine β_1, β_2 from Eq. (3.3) (for both over-damped and under-damped cases) and then calculated $a_{1,0}$, $a_{2,0}$ as well as a_I, φ_I up to second approximation.

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