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EXTREMUM PROPERTIES OF THE NEW ELLIPSOID

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Abstract. For a convex body K in $\mathbb{R}^n$, Lutwak, Yang and Zhang defined a new ellipsoid $\Gamma_{-2}K$, which is the dual analog of the Legendre ellipsoid. In this paper, we prove the following two results: (i) For any origin-symmetric convex body K, there exist an ellipsoid E and a parallelotope P such that $\Gamma_{-2}E \supseteq \Gamma_{-2}K \supseteq \Gamma_{-2}P$ and $V(E) = V(K) = V(P)$; (ii) For any convex body K whose John point is at the origin, then there exists a simplex T such that $\Gamma_{-2}K \supseteq \Gamma_{-2}T$ and $V(K) = V(T)$.

1. Introduction

For each convex subset in $\mathbb{R}^n$, it is well-known that there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the the moment of inertia of the convex set are the same about every 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the Lengendre ellipsoid of the convex set. The Lengendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics. See ([4, 5, 10]) for historical references.

It has slowly come to be recognized that along side the Brunn-Minkowski theory there is a dual theory. The Lengendre ellipsoid (and Binet ellipsoid) is an object of this dual Brunn-Minkowski theory. A nature question is whether there is a dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory. Applying the L_p-curvature theory ([7,8]), Lutwak, Yang and Zhang demonstrated the existence of precisely this dual object. Further, some beautiful and deep properties for this dual analog of the Legendre ellipsoid have been discovered ([9], see also [6]).

An often used fact in convex geometry is that associated with each convex body K is a unique ellipsoid JK of maximal volume that is contained in K. The ellipsoid is called the John ellipsoid of K and the center of this ellipsoid is called John point of K.

Let $V(K)$ denote the n-dimensional volume of the convex body K in $\mathbb{R}^n$. Let B_n denote the unit ball in $\mathbb{R}^n$ and ω_n the volume of B_n. For each convex body K, let $\Gamma_{-2}K$ denote the new ellipsoid which is defined by Lutwak, Yang and Zhang. In [9], the authors proved the following two theorems for the new ellipsoid.

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Theorem A. *Let K be an origin-symmetric convex body. Then*

$$2^{-n}\omega_n V(K) \leq V(\Gamma_{-2}K) \leq V(K).$$

Equality on the left-hand side holds if and only if K is a parallelotope and equality on the right-hand side holds if and only if K is an ellipsoid.

Theorem B. *Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, then*

$$V(\Gamma_{-2}K) \geq \frac{n!\omega_n}{n^{n/2}(n+1)^{(n+1)/2}} V(K),$$

with equality if and only if K is a simplex.

The aim of this paper is to study the new ellipsoid further. We give two extremal properties of the new ellipsoid. Our main results are the following two theorems:

Theorem 1. *For any origin-symmetric convex body K, there exist an ellipsoid E and a parallelotope P such that*

$$\Gamma_{-2}E \supseteq \Gamma_{-2}K \supseteq \Gamma_{-2}P \qquad \text{and} \qquad V(E) = V(K) = V(P).$$

Equality on the left-hand side holds if and only if K is an ellipsoid and equality on the right-hand side holds if and only if K is a parallelotope.

Theorem 2. *Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, then there exists a simplex T such that*

$$\Gamma_{-2}K \supseteq \Gamma_{-2}T \qquad \text{and} \qquad V(K) = V(T),$$

with equality if and only if K is a simplex.

As usual, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Let K be a nonempty compact convex body in $\mathbb{R}^n$, the support function h_K of K is defined by ([1,11])

$$h_K(u) = max\{u \cdot x : x \in K\}, u \in S^{n-1},$$

where $u \cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

For a compact subset L of $\mathbb{R}^n$, which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$ ([1,11])

$$\rho(L, u) = \max\{\lambda > 0 : \lambda u \in L\}.$$

For each convex body K, the new ellipsoid $\Gamma_{-2}K$ was defined by ([9])

$$\rho^{-2}_{\Gamma_{-2}K}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^2 dS_2(K, v), \tag{1}$$

for all $u \in S^{n-1}$, where $S_2(K, \cdot)$ denotes the L_2-surface measure.

It was shown in [7] that $S_2(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure S_K and that the Radon-Nikodym derivative

$$\frac{dS_2(K, \cdot)}{dS_K} = \frac{1}{h_K}.$$

Thus, if P is a polytope whose faces have outer unit normals $u_1, \ldots, u_N$, and a_i denotes the area ($(n-1)$-dimensional volumes) of the face with outer normal u_i and h_i denotes the distance from the origin to this face, then the measure $S_2(P, \cdot)$ is concentrated at the points $u_1, \ldots, u_N \in S^{n-1}$ and $S_2(P, u_i) = a_i/h_i$. Thus, for the polytope P, we have for $u \in S^{n-1}$

$$\rho_{\Gamma_{-2}P}^{-2}(u) = \frac{1}{V(P)} \sum_{i=1}^{N} (u \cdot u_i)^2 \frac{a_i}{h_i}. \tag{2}$$

Suppose K is a convex body that contains the origin in its interior. In [9], the authors proved that $\Gamma_{-2}K$ is affine invariant, i.e., if $\phi \in GL(n)$, then

$$\Gamma_{-2}(\phi K) = \phi \Gamma_{-2} K. \tag{3}$$

Since $\Gamma_{-2}B_n = B_n$, it follows from (3) that if E is an ellipsoid centered at the origin, then

$$\Gamma_{-2}E = E. \tag{4}$$

To prove of Theorem 1 and Theorem 2, we need the following lemmas.

Lemma 1.(John [2]) *Each convex body K contains an unique ellipsoid JK of maximal volume. This ellipsoid is the unit ball B_n if and only if the following conditions are satisfied: $B_n \subset K$ and there are contact points $\{u_i\}_1^m$ and positive numbers $\{c_i\}_1^m$ so that*

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{5}$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the span of u_i and I_n is the identity on $\mathbb{R}^n$.

The condition (5) shows that the u_i behave like an orthonormal basis to the extent that, for each $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^{m} c_i (x \cdot u_i)^2. \tag{6}$$

The equality of the traces in (5) shows that

$$\sum_{i=1}^{m} c_i = n. \tag{7}$$

Lemma 2. *Let C be a cube centered at the origin in $\mathbb{R}^n$. Then*

$$\Gamma_{-2}C = JC.$$

Proof. Without loss of generality, let C be $[-1,1]^n$ in $\mathbb{R}^n$. For C the John ellipsoid is B_n. The contact points are the standard basis vectors $(e_1,\ldots,e_n)$ of $\mathbb{R}^n$ and their negatives, and they satisfy

$$\sum_{i=1}^{m} e_i \otimes e_i = I_n.$$

That is, one can take all the weights c_i equal to 1 in (6).

Hence, it is sufficient to prove that the new ellipsoid is B_n. By the definition (2) and (6), we have for $u \in S^{n-1}$

$$\begin{aligned}\rho^{-2}_{\Gamma_{-2}C}(u) &= \frac{1}{V(C)}\sum_{i=1}^{2n}(u\cdot e_i)^2\frac{a_i}{h_i}\\ &= \frac{2}{2^n}\sum_{i=1}^{n}(u\cdot e_i)^2 2^{n-1}\\ &= \|u\|^2 = 1.\end{aligned}$$

Hence $\Gamma_{-2}C = B_n$, the proof of lemma 2 is completed.

Lemma 3.(Fejes Tóth [13]) *Let T be a simplex in $\mathbb{R}^n$ with inscribed ball radius r. Then*

$$V(T) \geq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n,$$

with equality if and only if T is a regular simplex.

Lemma 4. *Let T be a simplex in $\mathbb{R}^n$ that contains the origin in its interior. Then*

$$\Gamma_{-2}T = JT.$$

Proof. Since both $\Gamma_{-2}T$ and JT are affine invariant, it suffices to prove that $\Gamma_{-2}T = JT$ holds for a regular simplex. It is easy to verify that the John ellipsoid of a regular simplex is its inscribed ball.

Without loss of generality, we may assume that T is a regular simplex whose inscribed ball is B_n. Let $u_1,\ldots,u_{n+1}$ denote the outer unit normals and S the area of the face of T. By (6), we have for each $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^{n+1} c_i(x\cdot u_i)^2, \qquad \sum_{i=1}^{n+1} c_i = n.$$

Take u_i for x, and notice that $(u_i \cdot u_j) = -\dfrac{1}{n}$ for $i \neq j$.

It follows that

$$1 = \sum_{j=1}^{n+1} c_j (u_j \cdot u_i)^2 = \frac{1}{n^2}\left(\sum_{j=1}^{n+1} c_j - c_i\right) + c_i.$$

Hence

$$c_i = \frac{n}{n+1}.$$

By (2), we have

$$\begin{aligned}\rho_{\Gamma_{-2}T}^{-2}(u) &= \frac{1}{V(T)}\sum_{i=1}^{n+1}(u \cdot u_i)^2 S \\ &= \frac{n}{n+1}\sum_{i=1}^{n+1}(u \cdot u_i)^2 \\ &= \|u\|^2 = 1.\end{aligned}$$

Hence $\Gamma_{-2}T = B_n$, the proof of lemma 4 is completed.

Now we give the proofs of theorems.

Proof of Theorem 1. We first prove the left inclusion. Let

$$V(\Gamma_{-2}K) = \lambda V(K).$$

From Theorem A, we have

$$\frac{\omega_n}{2^n} \leq \lambda \leq 1. \tag{8}$$

Now put

$$E = \lambda^{-\frac{1}{n}}\Gamma_{-2}K,$$

we obtain

$$V(E) = V(\lambda^{-\frac{1}{n}}\Gamma_{-2}K) = \lambda^{-1}V(\Gamma_{-2}K) = V(K).$$

Since $\lambda^{-\frac{1}{n}} \geq 1$, we have

$$\Gamma_{-2}E = \Gamma_{-2}(\lambda^{-\frac{1}{n}}\Gamma_{-2}K) = \lambda^{-\frac{1}{n}}\Gamma_{-2}K \supseteq \Gamma_{-2}K.$$

Now we prove the right inclusion of Theorem 1.

Since $\Gamma_{-2}K$ is an ellipsoid, there exists a $\phi \in SL(n)$ such that $\phi(\Gamma_{-2}K)$ is a ball, that is,

$$\phi(\Gamma_{-2}K) = \left(\frac{V(\Gamma_{-2}K)}{\omega_n}\right)^{\frac{1}{n}} B_n. \tag{9}$$

Let C be the cube centered at the origin with the side length $V(K)^{\frac{1}{n}}$, so $V(C) = V(K)$. By Lemma 2, we have

$$\Gamma_{-2}C = JC = \frac{1}{2}V(K)^{\frac{1}{n}} B_n. \tag{10}$$

From (9),(10) and the left-hand side inequality of Theorem A, we have

$$\phi(\Gamma_{-2}K) \supseteq \Gamma_{-2}C,$$

that is,

$$\phi(\Gamma_{-2}K) \supseteq \phi[\Gamma_{-2}(\phi^{-1}C)].$$

It follows that

$$\Gamma_{-2}K \supseteq \Gamma_{-2}(\phi^{-1}C).$$

Let $P = \phi^{-1}C$, then

$$\Gamma_{-2}K \supseteq \Gamma_{-2}(P) \qquad \text{and} \qquad V(K) = V(C) = V(P).$$

The proof of Theorem 1 is completed.

Proof of Theorem 2. As in the proof of Theorem 1, there exists a $\phi \in SL(n)$ such that $\phi(\Gamma_{-2}K)$ is a ball, that is

$$\phi(\Gamma_{-2}K) = \left(\frac{V(\Gamma_{-2}K)}{\omega_n}\right)^{\frac{1}{n}} B_n. \tag{11}$$

Construct a regular simplex $T^{'}$ with inscribed ball radius $r = \left(\frac{n!V(K)}{n^{n/2}(n+1)^{(n+1)/2}}\right)^{\frac{1}{n}}$. By Lemma 4, we know $V(T^{'}) = V(K)$.

By Lemma 3, we have

$$\Gamma_{-2}T = JT^{'} = rB_n = \left(\frac{n!V(K)}{n^{n/2}(n+1)^{(n+1)/2}}\right)^{\frac{1}{n}} B_n. \tag{12}$$

According to the Theorem B, from (11) and (12), we infer that

$$\phi(\Gamma_{-2}K) \supseteq \Gamma_{-2}T^{'},$$

that is,

$$\Gamma_{-2}K \supseteq \Gamma_{-2}(\phi^{-1}T^{'}).$$

Let $T = \phi^{-1}T^{'}$, then

$$\Gamma_{-2}K \supseteq \Gamma_{-2}(T) \qquad \text{and} \qquad V(K) = V(T^{'}) = V(T).$$

The proof of Theorem 2 is completed.

The idea of this paper origins from Schneider's excellent survey [12] and Jonasson [3].

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