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AN INTERIOR INVERSE PROBLEM FOR STURM-LIOUVILLE OPERATORS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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Abstract. In this paper, we consider the inverse problem for Sturm-Liouville operators with eigenparameter dependent boundary conditions and show that the potential $q(x)$ and coefficients $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ and $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the eigenparameter dependent boundary conditions can be uniquely determined by a set of values of eigenfunctions at some interior point and parts of two spectra.

1. Introduction

In 1929, Ambartsumyan [1] firstly considered the inverse problem for Sturm-Liouville problems. Since 1929, one of this kind of inverse problems was discussed by many authors (see [1], [7]-[18]). Mochizuki and Trooshin [12] studied the inverse problem for interior spectral data of Sturm-Liouville operators on the finite interval $[0,1]$ and showed that a set of values of eigenfunctions at some interior point and parts of two spectra can uniquely determine the potential $q(x)$. Yang (C. F.) and Yang (X. P.) [13] discussed the inverse problem for Sturm-Liouville operators with discontinuous boundary conditions and proved that the spectral data of parts of two spectra and some information on eigenfunctions at some interior point of the interval $(0,\pi)$ is sufficient to determine the potential $q(x)$.

Consider the following Sturm-Liouville operator L satisfying (1.1)-(1.3)

$$Ly = -y'' + q(x)y = \lambda y, \tag{1.1}$$

with boundary conditions

$$(a_1\lambda + b_1)y(0,\lambda) - (c_1\lambda + d_1)y'(0,\lambda) = 0 \tag{1.2}$$

or

$$y'(0,\lambda) = hy(0,\lambda) \ or \ y(0,\lambda) = 0, \tag{1.2'}$$

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and

$$(a_2\lambda + b_2)y(\pi,\lambda) - (c_2\lambda + d_2)y'(\pi,\lambda) = 0. \tag{1.3}$$

where $a_k, b_k, c_k, d_k \in \mathbf{R}, (-1)^k\delta_k = a_kd_k - b_kc_k < 0 (k = 1,2)$, $q(x)$ is a real-valued function and in $L^2[0,\pi]$.

Sturm-Liouville problem with eigenparameter dependent boundary conditions is interesting in Engineering, Physics, Mathematics, etc (see [2]-[10]). In 1977, Fulton [6] studied the Sturm-Liouville problem (1.1), (1.2′), (1.3) and obtained the spectral theory of this kind of problems. Binding, Browne and Seddighi [7] considered the Sturm-Liouville problem (1.1)-(1.3) and got the oscillation, comparison results and asymptotic estimates. Using nodal points as spectral data, Browne and Sleeman [8] discussed the inverse nodal problem for the boundary value problem (1.1), (1.2′), (1.3) and showed that a dense set of nodal points of eigenfunctions for the boundary value problem (1.1), (1.2′), (1.3) is sufficient to determine the potential $q(x)$ and coefficient h of the boundary condition. Guliyev [9] found the regularized trace formula for the Sturm-Liouville problem (1.1)-(1.3). Wang, Yang and Huang [10] discussed the half inverse problem for the Sturm-Liouville problem (1.1)-(1.3) and showed that if $q(x)$ is prescribed on $[\pi/2,\pi]$ and $q - \tilde{q} \in W_2^6[0,\pi]$, then the potential $q(x)$ on the whole interval $[0,\pi]$ and coefficient $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the eigenparameter dependent boundary condition are uniquely determined by one spectrum.

In this paper, using Mochizuki and Trooshin's method, we discuss the interior inverse problem for Sturm-Liouville problem (1.1)-(1.3) and show that some information on eigenfunctions at some interior point of the interval $(0,\pi)$ and parts of two spectra are sufficient to determine the potential $q(x)$ and coefficients $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ and $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the eigenparameter dependent boundary conditions.

The following two lemmas are important for us to show the main theorems.

Lemma 1.1. ([7, Theorem 4.2], [9, Theorem]) *Let $\lambda_n (n = 0,1,2,\dots)$ be spectrum of the Sturm-Liouville problem* (1.1)−(1.3)*, then λ_n is real and simple and satisfies*

$$\begin{aligned} &\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty, \\ &\sqrt{\lambda}_n = n(1 + O(\tfrac{1}{n})). \end{aligned} \tag{1.4}$$

Suppose that $\varphi(x), \theta(x)$ are two fundamental solutions of the equation (1.1) satisfying

$$\varphi(0) = 1, \varphi'(0) = 0, \theta(0) = 0 \text{ and } \theta'(0) = 1,$$

respectively, then solution of the equation (1.1) satisfying (1.2) is

$$y(x,\lambda) = (c_1\lambda + d_1)\varphi(x) + (a_1\lambda + b_1)\theta(x).$$

By transformation, we have

Lemma 1.2. *Solution of the equation* (1.1) *satisfying* (1.2) *is*

$$y(x,\lambda) = (c_1\lambda+d_1)[\cos\sqrt{\lambda}x+\int_0^x A(x,t)\cos\sqrt{\lambda}t\,dt]$$
$$+(a_1\lambda+b_1)[\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}\int_0^x B(x,t)\sin\sqrt{\lambda}t\,dt], \tag{1.5}$$

where the kernel $A(x,t)$ *satisfies*

$$\frac{\partial^2 A(x,t)}{\partial x^2} - q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2},$$

where $q(x) = 2\frac{d}{dx}A(x,x)$, $A(0,0) = h$, $\frac{\partial A(x,t)}{\partial t}|_{t=0} = 0$. *The kernel* $B(x,t)$ *satisfies*

$$\frac{\partial^2 B(x,t)}{\partial x^2} - q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2},$$

where $q(x) = 2\frac{d}{dx}B(x,x)$, $B(x,0) = 0$.

2. Main results

Consider another Sturm-Liouville operator $\tilde{L}$

$$\tilde{L}y = -y'' + \tilde{q}(x)y = \lambda y, \tag{2.1}$$

with boundary conditions

$$(\tilde{a}_1\lambda+\tilde{b}_1)y(0,\lambda) - (\tilde{c}_1\lambda+\tilde{d}_1)y'(0,\lambda) = 0, \tag{2.2}$$
$$(\tilde{a}_2\lambda+\tilde{b}_2)y(\pi,\lambda) - (\tilde{c}_2\lambda+\tilde{d}_2)y'(\pi,\lambda) = 0, \tag{2.3}$$

where $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k, \tilde{d}_k, \in \mathbf{R}$, $(-1)^k\tilde{\delta}_k = \tilde{a}_k\tilde{d}_k - \tilde{b}_k\tilde{c}_k < 0 (k = 1,2)$, $\tilde{q}(x)$ is a real-valued function in $L^2[0,\pi]$.

By virtue of [10], we obtain the following Lemma 2.1, which plays an important role in the proof of Theorem 2.2.

Lemma 2.1. ([10, Theorem]) *Let* $\{\lambda_n\}(n\geq 0)$ *be real and simple spectrum of the Sturm-Liouville problem* (1.1)-(1.3) *and* $\{\tilde{\lambda}_n\}(n\geq 0)$ *be real and simple spectrum of the Sturm-Liouville problem* (2.1), (2.2), (1.3), *respectively and* $q-\tilde{q} \in W_2^6[0,\pi]$. *If* $\lambda_n = \tilde{\lambda}_n (n\geq 0)$ *and* $q(x) = \tilde{q}(x)$ *on* $[\pi/2,\pi]$, *then*

$$q(x) = \tilde{q}(x) \ \ a.e. \ \ on \ [0,\pi],$$

and

$$\frac{\tilde{a}_1\lambda+\tilde{b}_1}{\tilde{c}_1\lambda+\tilde{d}_1} = \frac{a_1\lambda+b_1}{c_1\lambda+d_1}, \ \forall\lambda\in\mathbf{C},$$

where $q(x)$, $\tilde{q}(x)$ *are real-valued functions in* $L^2[0,\pi]$.

Next, we present the main results in this paper. When $b = \pi/2$, we get the following uniqueness Theorem 2.2.

Theorem 2.2. *Let* $\{\lambda_n\}$ *and* $\{\tilde{\lambda}_n\}$ *be a spectrum of both Sturm-Liouville problem* (1.1)-(1.3) *and Sturm-Liouville problem* (2.1)-(2.3)*, respectively and* $q-\tilde{q} \in W_2^6[0,\pi]$*. If for any* n $(n = 0,1,2,\dots)$

$$\lambda_n = \tilde{\lambda}_n \text{ and } \frac{y_n'(\pi/2,\lambda_n)}{y_n(\pi/2,\lambda_n)} = \frac{\tilde{y}'_n(\pi/2,\tilde{\lambda}_n)}{\tilde{y}_n(\pi/2,\tilde{\lambda}_n)}, \tag{2.4}$$

then

$$q(x) = \tilde{q}(x) \quad \text{a.e. on } [0,\pi]$$

and

$$\frac{\tilde{a}_1\lambda+\tilde{b}_1}{\tilde{c}_1\lambda+\tilde{d}_1} = \frac{a_1\lambda+b_1}{c_1\lambda+d_1} \text{ and } \frac{\tilde{a}_2\lambda+\tilde{b}_2}{\tilde{c}_2\lambda+\tilde{d}_2} = \frac{a_2\lambda+b_2}{c_2\lambda+d_2} (\forall \lambda \in \mathbf{C}),$$

where $y_n(x,\lambda_n)$ *is an eigenfunction of* λ_n *and* $\tilde{y}_n(x,\tilde{\lambda}_n)$ *is an eigenfunction of* $\tilde{\lambda}_n$.

When $b \neq \pi/2$, the uniqueness theorem of $q(x)$ can be obtained from a part of the second spectrum and some information on eigenfunctions at the point $b \in (0,\pi)$.

Let $m(n)$ be a subsequence of natural numbers such that

$$m(n) = \frac{n}{\sigma}(1+\varepsilon_n),\ 0 < \sigma \le \pi,\ \varepsilon_n \to 0. \tag{2.5}$$

Lemma 2.3. *Let* $m(n)$ *be a subsequence of natural numbers satisfying* (2.5)*,* $b \in (0,\pi/2)$ *be such that* $\sigma > \frac{2b}{\pi}$ *and* $q-\tilde{q} \in W_2^6[0,\pi]$*. If for any* $n(n = 0,1,2,\dots)$*,*

$$\lambda_{m(n)} = \tilde{\lambda}_{m(n)} \text{ and } \frac{y'_{m(n)}(b,\lambda_{m(n)})}{y_{m(n)}(b,\lambda_{m(n)})} = \frac{\tilde{y}'_{m(n)}(b,\tilde{\lambda}_{m(n)})}{\tilde{y}_{m(n)}(b,\tilde{\lambda}_{m(n)})}, \tag{2.6}$$

then

$$q(x) = \tilde{q}(x) \quad \text{a.e. on } [0,b] \text{ and } \quad \frac{\tilde{a}_1\lambda+\tilde{b}_1}{\tilde{c}_1\lambda+\tilde{d}_1} = \frac{a_1\lambda+b_1}{c_1\lambda+d_1} (\forall \lambda \in \mathbf{C}).$$

Let $l(n)$ and $r(n)$ be a subsequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1+\varepsilon_{1,n}),\ 0 < \sigma_1 \le \pi,\ \varepsilon_{1,n} \to 0, \tag{2.7}$$

$$r(n) = \frac{n}{\sigma_2}(1+\varepsilon_{2,n}),\ 0 < \sigma_2 \le \pi,\ \varepsilon_{2,n} \to 0 \tag{2.8}$$

and let μ_n be the eigenvalues of the problem (1.1),(1.2) and (2.9) and $\tilde{\mu}_n$ be the eigenvalues of the problem (2.1), (2.2) and (2.9).

$$(a_3\lambda+b_3)y(\pi,\lambda) - (c_3\lambda+d_3)y'(\pi,\lambda) = 0, \tag{2.9}$$

where $\delta_3 = a_3d_3 - b_3c_3 < 0, \delta_3 \neq \delta_2$.

Using Mochizuki and Trooshin's method, from Lemma 2.3 and Theorem 2.2, we will prove that the following Theorem 2.4 holds.

Theorem 2.4. *Let $l(n)$ and $r(n)$ be subsequence of natural numbers satisfying* (2.7) *and* (2.8), *respectively, and $b \in (\pi/2, \pi)$ be such that $\sigma_1 > \frac{2b}{\pi} - 1$, $\sigma_2 > 2 - \frac{2b}{\pi}$ and $q - \tilde{q} \in W_2^6[0,\pi]$. If for any n $(n = 0, 1, 2, \ldots)$,*

$$\lambda_n = \tilde{\lambda}_n,\ \mu_{l(n)} = \tilde{\mu}_{l(n)} \ \textit{and}\ \frac{y'_{r(n)}(b, \lambda_{r(n)})}{y_{r(n)}(b, \lambda_{r(n)})} = \frac{\tilde{y}'_{r(n)}(b, \tilde{\lambda}_{r(n)})}{\tilde{y}_{r(n)}(b, \tilde{\lambda}_{r(n)})},$$

then

$$q(x) = \tilde{q}(x) \quad \textit{a.e. on } [0,\pi]$$

and

$$\frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1} \ \textit{and}\ \frac{\tilde{a}_2\lambda + \tilde{b}_2}{\tilde{c}_2\lambda + \tilde{d}_2} = \frac{a_2\lambda + b_2}{c_2\lambda + d_2} (\forall \lambda \in \mathbf{C}).$$

3. Proofs of main results

In this section, we present the proofs of main results in this paper.

Proof of Theorem 2.2. We give the proof of Theorem 2.2 by two steps.

Step 1: From the assumptions of Theorem 2.2, similar to the proof of Lemma 2.1, we have

$$q(x) = \tilde{q}(x) \quad \text{a.e. on } [0,\pi/2] \text{ and } \quad \frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1} (\forall \lambda \in \mathbf{C}). \tag{3.1}$$

Step 2: Consider the following supplementary problem

$$\begin{aligned} L_1 y = -y'' + q_1(x) y = \lambda y, \\ q_1(x) = q(\pi - x),\ x \in [0,\pi], \end{aligned} \tag{3.2}$$

with the boundary conditions

$$(a_2\lambda + b_2) y(0,\lambda) - (c_2\lambda + d_2) y'(0,\lambda) = 0, \tag{3.3}$$

$$(a_1\lambda + b_1) y(\pi,\lambda) - (c_1\lambda + d_1) y'(\pi,\lambda) = 0. \tag{3.4}$$

and

$$\begin{aligned} \tilde{L}_1 y = -y'' + \tilde{q}_1(x) y = \lambda y, \\ \tilde{q}_1(x) = \tilde{q}(\pi - x),\ x \in [0,\pi], \end{aligned} \tag{3.5}$$

with the boundary conditions

$$(\tilde{a}_2\lambda + \tilde{b}_2) y(0,\lambda) - (\tilde{c}_2\lambda + \tilde{d}_2) y'(0,\lambda) = 0, \tag{3.6}$$

$$(\tilde{a}_1\lambda + \tilde{b}_1) y(\pi,\lambda) - (\tilde{c}_1\lambda + \tilde{d}_1) y'(\pi,\lambda) = 0. \tag{3.7}$$

Repeating the Step 1 for the supplementary problem, we get

$$q(x) = \tilde{q}(x) \quad \text{a.e. on } [\pi/2, \pi] \text{ and } \quad \frac{\tilde{a}_2\lambda + \tilde{b}_2}{\tilde{c}_2\lambda + \tilde{d}_2} = \frac{a_2\lambda + b_2}{c_2\lambda + d_2} (\forall \lambda \in \mathbf{C}). \tag{3.8}$$

By virtue of (3.1) and (3.8), this yields

$$q(x) = \tilde{q}(x) \quad \text{a.e. on} \quad [0, \pi]$$

and

$$\frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1} \text{ and } \frac{\tilde{a}_2\lambda + \tilde{b}_2}{\tilde{c}_2\lambda + \tilde{d}_2} = \frac{a_2\lambda + b_2}{c_2\lambda + d_2} (\forall \lambda \in \mathbf{C}).$$

Therefore, Theorem 2.2 is proved.

Next, we show that Lemma 2.3 holds.

Proof of Lemma 2.3. Multiplying (2.1) by y, (1.1) by $\tilde{y}$, subtracting and integrating from 0 to b, we obtain

$$\begin{aligned} G(\lambda) &:= \int_0^b Q(x) y(x,\lambda) \tilde{y}(x,\lambda) dx \\ &= [\tilde{y}(x,\lambda) y'(x,\lambda) - y(x,\lambda)\tilde{y}'(x,\lambda)]|_0^b. \end{aligned} \tag{3.9}$$

From (1.2), (2.2) and the assumption

$$\frac{y'_{m(n)}(b, \lambda_{m(n)})}{y_{m(n)}(b, \lambda_{m(n)})} = \frac{\tilde{y}'_{m(n)}(b, \tilde{\lambda}_{m(n)})}{\tilde{y}_{m(n)}(b, \tilde{\lambda}_{m(n)})},$$

we get

$$G(\lambda_m(n)) = 0, \ n \in N.$$

Next, we will show $G(\lambda) = 0, \ \forall \lambda \in \mathbf{C}$.

From (3.9), we see that the entire function $G(\lambda)$ is a function of exponential type $\leq 2b$ and we have

$$|G(\lambda)| \leq M e^{2br|\sin\theta|}, \tag{3.10}$$

where M is a positive constant, $\lambda = re^{i\theta}$.

Define the indicator of function $G(\lambda)$ by

$$h(\theta) = \limsup_{\lambda \to +\infty} \frac{\ln|G(re^{i\theta})|}{r}. \tag{3.11}$$

Since $|Im\lambda| = r|\sin\theta|$, $\theta = arg\lambda$, from (3.10) and (3.11), we get

$$h(\theta) \leq 2b|\sin\theta|. \tag{3.12}$$

Let $n(r)$ be the number of zeros of $G(\lambda)$ in the disk $|\lambda| \leq r$. From the assumption of Lemma 2.3 and the asymptotic form (1.4) of the eigenvalues λ_n, we obtain

$$n(r) \geq 2 \sum_{\frac{n}{\sigma}[1+O(1/n)]<r} 1 \geq 2\sigma r[1 + \varepsilon(r)], r \to \infty, \tag{3.13}$$

where $\varepsilon(r) \to 0, r \to \infty$, $[x]$ is the integer part of x.

For the case $\sigma > \frac{2b}{\pi}$,

$$\liminf_{n\to\infty} \frac{n(r)}{r} \geq 2\sigma > \frac{4b}{\pi} \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.14}$$

According to [19], for any entire function $G(\lambda)$ of exponential type, not identically zero, then

$$\liminf_{n\to\infty} \frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.15}$$

The inequalities (3.14) and (3.15) imply that $G(\lambda) = 0,\ \forall \lambda \in \mathbf{C}$.

Similar to the proof of the Lemma 2.1, we have

$$q(x) = \tilde{q}(x) \quad \text{a.e. on } [0,b] \text{ and } \quad \frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}.$$

This completes the proof of Lemma 2.3.

Now, we prove that Theorem 2.4 is valid.

Proof of Theorem 2.4. From

$$\lambda_{r(n)} = \tilde{\lambda}_{r(n)},\ and\ \frac{y'_{r(n)}(b,\lambda_{r(n)})}{y_{r(n)}(b,\lambda_{r(n)})} = \frac{\tilde{y}'_{r(n)}(b,\tilde{\lambda}_{r(n)})}{\tilde{y}_{r(n)}(b,\tilde{\lambda}_{r(n)})},$$

where $r(n)$ satisfies (2.8) and $\sigma_2 > 2 - \frac{2b}{\pi}$, according to Lemma 2.3, we get

$$q(x) = \tilde{q}(x), \quad \text{a.e. on } [b,\pi] \text{ and } \quad \frac{\tilde{a}_2\lambda + \tilde{b}_2}{\tilde{c}_2\lambda + \tilde{d}_2} = \frac{a_2\lambda + b_2}{c_2\lambda + d_2}(\forall \lambda \in \mathbf{C}) \tag{3.16}$$

Let $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ be the eigenfunctions of Sturm-Liouville problems (1.1)−(1.3) and (2.1)−(2.3), corresponding to eigenvalue λ_n, respectively. Since the eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ satisfy the same boundary condition at π and $\tilde{q}(x) = q(x)$, a.e. on $[b,\pi]$, we obtain

$$y_n(x,\lambda_n) = \alpha_n \tilde{y}_n(x,\lambda_n), x \in [b,\pi],\ n \in \mathbf{N}, \tag{3.17}$$

where α_n are constants.

From (3.9), (3.17), (1.2) and (2.2), we obtain

$$G(\lambda_n) = 0,\ n \in \mathbf{N}$$

and

$$G(\mu_{l(n)}) = 0,\ n \in \mathbf{N},$$

where λ_n and $\mu_{l(n)}$ satisfy (1.4).

Let us count the number of λ_n and $\mu_{l(n)}$ located inside the disc of radius r (sufficiently large r). We see that there are $1+2r[1+o(1)]$ of λ_n and $1+2r\sigma_1[1+o(1)]$ of $\mu_{l(n)}$ located inside the disc of radius r. Therefore

$$n(r)=2+2r[1+\sigma_1+o(1)].$$

Hence

$$\lim_{n\to\infty}\frac{n(r)}{r}=2(\sigma_1+1).$$

Considering the condition $\sigma_1>\frac{2b}{\pi}-1$, we get

$$\lim_{n\to\infty}\frac{n(r)}{r}\geq 2\sigma_1>\frac{4b}{\pi}\geq\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta. \tag{3.18}$$

According to [19], for any entire function $G(\lambda)$ of exponential type, not identically zero, we see the following inequality (3.19) holds.

$$\liminf_{n\to\infty}\frac{n(r)}{r}\leq\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta. \tag{3.19}$$

The inequalities (3.18) and (3.19) imply that

$$G(\lambda)=0,\ \forall\lambda\in\mathbf{C}. \tag{3.20}$$

From (3.20), we can show that

$$q(x)=\tilde{q}(x)\quad\text{a.e. on }[0,b]\text{ and }\quad\frac{\tilde{a}_1\lambda+\tilde{b}_1}{\tilde{c}_1\lambda+\tilde{d}_1}=\frac{a_1\lambda+b_1}{c_1\lambda+d_1}. \tag{3.21}$$

From (3.15) and (3.21), we have

$$q(x)=\tilde{q}(x)\quad\text{a.e. on }[0,\pi]$$

and

$$\frac{\tilde{a}_1\lambda+\tilde{b}_1}{\tilde{c}_1\lambda+\tilde{d}_1}=\frac{a_1\lambda+b_1}{c_1\lambda+d_1}\text{ and }\frac{\tilde{a}_2\lambda+\tilde{b}_2}{\tilde{c}_2\lambda+\tilde{d}_2}=\frac{a_2\lambda+b_2}{c_2\lambda+d_2}(\forall\lambda\in\mathbf{C}).$$

Hence, the proof of Theorem 2.4 is completed.

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