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ALGEBRAIC ELEMENTARY OPERATORS ON $B(E)$

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Abstract. In this paper we have obtained a necessary and sufficient condition for generalized derivations to be algebraic on $B(E)$. Further some results on algebraicness of elementary operators are given.

1. Introduction

Let $B(E)$ be the algebra of all bounded linear operators on a Banach space E and $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be n-tuples of elements in $B(E)$. The elemenatary operator $R_{A,B}$ associated with A and B is the operator on $B(E)$ into itself defined by

$$R_{A,B}(X) = A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n \quad \text{for all } X \in B(E).$$

We say $A = (A_1, A_2, \ldots, A_n)$ is commuting family if $A_iA_j = A_jA_i$ for each $1 \le j \le n$. For A and B in $B(E)$, by $M_{A,B}$ we denote elementary multiplication operator defined by $M_{A,B}(X) = AXB$ for all $X \in B(E)$. This can also be seen as elementary operator of length one. For $A, B \in B(E)$, inner derivation δ_A on $B(E)$ into itself is defined by $\delta_A(X) = AX - XA$ and generalized derivation $\delta_{A,B}$ on $B(E)$ into itself is defined by $\delta_{A,B}(X) = AX - XB$ for all $X \in B(E)$. It is easy to see that generalized derivation and inner derivation are particular cases of elementary operators.

Definition 1.1. Let A be an associative algebra with identity. An elementary operator $E : A \to A$ is called algebraic if $p(E) = 0$ for some nonzero polynomial p.

Algebraic derivations are well-studied objects in the field of pure algebra. The first general result on algebraic elementary operators was obtained by S.A. Amitsure [1] who proved that an algebraic derivation on a simple ring of characteristic zero must be inner. Miers and Philips [8] have studied algebraic derivation in the setting of C^*-algebra. I. N. Herstein [5] has given the sufficient condition for an inner derivation to be algebraic on an associative algebra.

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Sanjay Kumar [9] has obtained a necessary and sufficient condition for an inner derivation to be algebraic on a separable Hilbert space by using spectral properties of inner derivation. For further work on this topic see [2, 3, 6, 7, 8] and references therein.

Theorem 1.1 ([5]). *If $a \in A$ is algebraic then δ_A is algebraic.*

Theorem 1.2 ([9]). *Let T be a bounded linear operator on a separable Hilbert space H. Then δ_T is algebraic if and only if T is algebraic.*

2. Main results

In this section we shall give a necessary condition for an elementary operator to be algebraic, and then a necessary and sufficient condition for a generalized derivation to be algebraic.

First we shall give some simple results about algebraic operators.

Proposition 2.1. *Let $B(E)$ be the algebra of all bounded linear operators on a Banach space E.*

(a) *If T is algebraic then T^2 is algebraic.*

(b) *If T_1 and T_2 are algebraic and $T_1T_2 = T_2T_1$ then*

  (i) *$T_1 + T_2$ is algebraic.*

  (ii) *T_1T_2 is algebraic.*

(c) *If T is algebraic then T^n is algebraic.*

Proof. (a) Suppose T is algebraic then $p(T) = 0$ for some nonzero polynomial p i.e. $\sum_{i=0}^{k} a_i T^i = 0$. Let degree of p be odd i.e. $k = 2m+1$.

Now $\sum_{i=0}^{2m+1} a_i T^i = 0$ i.e. $a_0 I + a_1 T + a_2 T^2 + \cdots + a_{2m+1} T^{2m+1} = 0$

$\implies (a_0 I + a_2 T^2 + a_4 T^4 + \cdots + a_{2m} T^{2m}) + (a_1 T + a_3 T^3 + \cdots + a_{2m+1} T^{2m+1}) = 0$

$\implies (a_0 I + a_2 T^2 + a_4 T^4 + \cdots + a_{2m} T^{2m}) = -(a_1 T + a_3 T^3 + \cdots + a_{2m+1} T^{2m+1})$.

It is easy to see by squaring both sides we get a nonzero polynomial q such that $q(T^2) = 0$.

If $n = 2m$, result follows similarly.

(b)(i) Let T_1 and T_2 be algebraic of degree m and n respectively. Suppose $p(T_1) = \sum_{i=0}^{m} a_i T_1^i = 0$ and $q(T_2) = \sum_{i=0}^{n} b_i T_2^i = 0$.

Now $p(T_1) = \sum_{i=0}^{m} a_i T_1^i = 0$

$\implies T_1^m = -1/a_m (a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1} T_1^{m-1})$.

Thus any power power of T_1 can be expressed as a linear combination of I, $T_1, T_1^2, \ldots, T_1^{m-1}$. Similarly any power of T_2 can be expressed as a linear combination of I, $T_2, T_2^2, \ldots, T_2^{n-1}$. Now

$$(T_1+T_2)^k = \sum_{i=0}^{k} \binom{k}{i} T_1^{k-i} T_2^i \quad (\because T_1T_2 = T_2T_1) = \sum_{p=0}^{m-1}\sum_{q=0}^{n-1} a_{pq} T_1^p T_2^q,$$

where a_{pq} are suitable constants.

Suppose r is a polynomial of degree mn i.e.$r(x) = \sum_{k=0}^{mn} c_k x^k$.

Now

$$\sum_{k=0}^{mn} c_k (T_1+T_2)^k = \sum_{k=0}^{mn} c_k \sum_{p=0}^{m-1}\sum_{q=0}^{n-1} a_{pq}^{(k)} T_1^p T_2^q$$

here $a_{pq}^{(k)}$ are suitable constants for $0 \le k \le mn$.

$$\sum_{k=0}^{mn} c_k (T_1+T_2)^k = \sum_{p=0}^{m-1}\sum_{q=0}^{n-1} \Big(\sum_{k=0}^{mn} a_{pq}^{(k)} c_k\Big) T_1^p T_2^q$$

Suppose $\sum_{k=0}^{mn} a_{pq}^{(k)} c_k = 0$ for each $0 \le p \le m-1$ and $0 \le q \le n-1$.
Since number of homogeneous equations is mn and number of constants is $mn+1$, which are treated as variable here. It follows that there is a nonzero solution in c_k's. Therefore there exist a polynomial r such that $r(T_1+T_2) = 0$.

(ii) Since $T_1 \pm T_2$ is algebraic, $(T_1 \pm T_2)^2$ is algebraic by **(a)**. Therefore $T_1T_2 = 1/4((T_1+T_2)^2 + (T_1-T_2)^2)$ is algebraic.

(c) By using result b(ii), it is easy to see that T^n is algebraic if T is algebraic. □

Remark 2.1. It is easy to see that if $T_1T_2 \ne T_2T_1$ then T_1+T_2 and T_1T_2 may not be algebraic.

Theorem 2.1. *Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be commuting families of elements in $B(E)$. Then the elementary operator $R_{A,B} = \sum_{i=1}^{n} A_i X B_i$ is algebraic if A_i and B_i are algebraic for each $1 \le i \le n$.*

Proof. First we shall prove it for $n = 1$.

Let $R_{A_1,B_1}(X) = A_1XB_1$, where A_1 and B_1 are algebraic. Suppose $p(A_1) = \sum_{i=0}^{m} a_i A_1^i = 0$, $a_m \ne 0$. Then

$$A_1^m = -\frac{1}{a_m}(a_0I + a_1A_1 + a_2A_1^2 + \cdots + a_{m-1}A_1^{m-1}).$$

It is easy to see that every power of A_1 can be expressed as a linear combination of I, $A_1, A_1^2, \ldots, A_1^{m-1}$. Similarly, suppose $q(B_1) = \sum_{i=0}^{n} b_i B_1^i = 0$, $b_n \ne 0$. Then $B_1^n = -1/b_n(b_0I + b_1B_1 + b_2B_1^2 +$

$\cdots + b_{n-1}B_1^{n-1})$. Therefore, every power of B_1 can be expressed as a linear combination of I, B_1, $B_1^2,\ldots, B_1^{n-1}$. Now

$$\begin{aligned} R^k(X) &= A_1^k X B_1^k \\ &= (\sum_{i=1}^{m} a_i A_1^i) X (\sum_{j=1}^{n} b_j B_1^j) \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} r_{ij} A_1^i X B_1^j, \end{aligned}$$

where r_{ij} are suitable constants for $0 \le i \le m-1$, $0 \le j \le n-1$.

Let $r(x) = \sum_{i=0}^{mn} c_k x^k$, then $r(R) = \sum_{i=0}^{mn} c_k R^k$. Now

$$\begin{aligned} r(R)(X) &= \sum_{k=0}^{mn} c_k \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} r_{ij}^{(k)} A_1^i X B_1^j \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (\sum_{k=0}^{mn} r_{ij}^{(k)} c_k) A_1^i X B_1^j, \end{aligned}$$

where $r_{ij}^{(k)}$ are suitable constants for $0 \le k \le mn$.

Now assume $\sum_{k=0}^{mn} r_{ij}^{(k)} c_k$ for each $0 \le i \le m-1$ and $0 \le j \le n-1$. Thus we have a system of homogeneous equation with mn equations and $mn+1$ constants c_k which are treated as variables. Therefore there exists a nonzero solution in c_k's and hence $r(R) = 0$.

Now suppose statement is true for $n = m$.

For $n = m+1$, let

$$\begin{aligned} R(X) &= \sum_{i=1}^{m+1} A_i X B_i \\ &= \sum_{i=1}^{m} A_i X B_i + A_{m+1} X B_{m+1} \\ &= R'(X) + M_{A_{m+1},B_{m+1}}, \end{aligned}$$

here $R'(X) = \sum_{i=1}^{m} A_i X B_i$. But $R' M_{A_{m+1}.B_{m+1}} = M_{A_{m+1},B_{m+1}} R'$ because A and B one commuting families. Therefore by Theorem 2.1 b(i) there exists a nonzero polynomial r such that $r(R) = 0$. □

Corollary 2.1. **(i)** *If R is an algebraic elementary operator then R^n is algebraic.*
(ii) *If R_1 and R_2 are algebraic with $R_1R_2 = R_2R_1$ then R_1R_2 is algebraic.*

Now we shall state a formula for n^{th} iterate of a generalized derivation which can be proved easily by using method of induction.

Lemma 2.1. *Let $\delta_{A,B}$ be generalized derivation on $B(E)$ into itself then*

$$\delta^n_{A,B}(X) = \sum_{k=0}^{n}((-1)^n\binom{n}{k})A^{n-k}XB^k.$$

Theorem 2.2. *The generalized derivation $\delta_{A,B}$ is algebraic if and only if A and B are algebraic.*

Proof. Let A and B are algebraic, then $\sum_{k=0}^{n}((-1)^n\binom{n}{k})A^{n-k}XB^k$ is algebraic by Theorem 2.1 because generalized derivation is a particular case of elementary operator of length 2.

Conversely, suppose $\delta_{A,B}$ is algebraic i.e. there exists a non-zero polynomial p of degree n such that $p(\delta_{A,B}) = \sum_{k=0}^{n} c_k\delta^k_{A,B} = 0$. We have $\delta^k_{A,B}(X) = \sum_{i=0}^{n}((-1)^i\binom{k}{i})A^{k-i}XB^i$. By consequence of Hahn-Banach theorem there exists a linear functional f such that $f(x) \neq 0$ for each nonzero x in E. Let $f \otimes x$ be a rank one operator on E defined by $(f \otimes x)(y) = f(y)x,\ x \in E$.

Now

$$\begin{aligned}\delta^k_{A,B}(f\otimes x) &= \sum_{i=0}^{k}\left((-1)^i\binom{k}{i}\right)A^{k-i}(f\otimes x)B^k,\\ \delta^k_{A,B}(f\otimes x)(x) &= \sum_{i=0}^{k}\left((-1)^i\binom{k}{i}\right)f(B^i x)A^{k-i}x\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{n} c_k\delta^k_{A,B}(f\otimes x)(x) &= \sum_{i=0}^{n} c_k\sum_{i=0}^{k}\left((-1)^i\binom{k}{i}\right)(f(B^i x)A^{k-i}x\\ &= \sum_{j=0}^{n}(b_jA^j)(x),\end{aligned}$$

where $b_j = \sum_{k=j}^{n}(-1)^k\binom{k}{j}c_kf(B^{k-j}x)$. In particular $b_n = (-1)^nc_nf(x) \neq 0$.

Since $\sum_{k=0}^{n} c_k\delta^k_{A,B}(X) = 0$ for all $X \in B(E)$, it follows that $\sum_{j=0}^{n} b_jA^j = 0$ i.e. A is algebraic. Further since an operator T is algebraic if and only if its transpose T^t is algebraic, it follows that $\delta_{A,B}$ is algebraic if and only if $\delta_{B,A}$ is algebraic.Thus by using same method as above we get B is algebraic. □

Theorem 2.3. *If the elementary operator $M_{A,B}$ is algebraic then either A or B is algebraic.*

Proof. Let $p(M_{A,B}) = \sum_{k=0}^{n} a_kM^k_{A,B} = 0, a_n \neq 0$. Then $\sum_{k=0}^{n} a_kM^k_{A,B}(X) = \sum_{k=0}^{n} a_kA^kXB^k = 0$ for all $X \in B(E)$.

Now suppose B is not algebraic. Therefore B is not nilpotent. Then there exists some nonzero vector $x \in E$ such that$B^n(x) \neq 0$. Now by using method similar as in proof of above theorem we get A is algebraic. □

Example 2.1. Let $B = I$, then $M_{A,B}(X) = AXB = AX$. Now $M_{A,B}$ is algebraic if and only if A is algebraic. If A is not algebraic then $M_{A,B}$ is not algebraic though $B = I$ is algebraic. This shows that converse of Theorem 2.1 is not true.

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