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SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS

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Abstract. In this paper, we introduce new subclasses of convex and starlike functions with respect to other points. The coefficient estimates for these classes are obtained.

1. Introduction

Let U be the class of functions which are analytic and univalent in the open unit disc $D = \{z : |z| < 1\}$ given by

$$w(z) = z + \sum_{k=1}^{n} b_k z^k$$

and satisfying the conditions

$$w(0) = 0, \ \ |w(z)| < 1, \ \ z \in D.$$

Let S denote the class of functions f which are analytic and univalent in D of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D. \tag{1.1}$$

Also let S_s^* be the subclass of S consisting of functions given by (1.1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. Ashwah and Thomas in [2] introduced another class namely the class S_c^* consisting of functions starlike with respect to conjugate points.

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Let S_c^* be the subclass of S consisting of functions given by (1.1) and satisfying the condition

$$Re\left\{\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right\}>0, \quad z\in D.$$

Motivated by S_s^*, many authors discussed the following class C_s of function convex with respect to symmetric points and its subclasses.

Let C_s be the subclass of S consisting of functions given by (1.1) and satisfying the condition

$$Re\left\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\right\}>0, \quad z\in D.$$

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of S_s^* denoted by $S_s^*(A,B)$.

Let $S_s^*(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz}, \quad -1\le B<A\le 1, z\in D.$$

Also let $S_c^*(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{(f(z)+\overline{f(\overline{z})})}\prec\frac{1+Az}{1+Bz}, \quad -1\le B<A\le 1, z\in D.$$

Let $C_s(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'}\prec\frac{1+Az}{1+Bz}, \quad -1\le B<A\le 1, z\in D.$$

Also let $C_c(A,B)$ be the class of functions of the form (1.1) and satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)+\overline{f(\overline{z})})'}\prec\frac{1+Az}{1+Bz}, \quad -1\le B<A\le 1, z\in D.$$

In this paper, we introduce the class $M_s(\alpha,A,B)$ consisting of analytic functions f of the form (1.1) and satisfying

$$\frac{2zf'(z)+2\alpha z^2f''(z)}{(1-\alpha)(f(z)-f(-z))+\alpha z(f(z)-f(-z))'}\prec\frac{1+Az}{1+Bz},$$
$$-1\le B<A\le 1, 0\le\alpha\le 1, z\in D.$$

We note that $M_s(0,A,B)=S_s^*(A,B)$ and $M_s(1,A,B)=C_s(A,B)$. Also introduce the class $M_c(\alpha,A,B)$ consisting of analytic functions f of the form (1.1) and satisfying

$$\frac{2zf'(z)+2\alpha z^2f''(z)}{(1-\alpha)(f(z)+\overline{f(\overline{z})})+\alpha z(f(z)+\overline{f(\overline{z})})'}\prec\frac{1+Az}{1+Bz},$$
$$-1\le B<A\le 1, 0\le\alpha\le 1, z\in D.$$

Note that $M_c(0,A,B) = S_c^*(A,B)$ and $M_c(1,A,B) = C_c(A,B)$.

By definition of subordination it follows that $f \in M_s(\alpha,A,B)$ if and only if

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)(f(z)-f(-z))+\alpha z(f(z)-f(-z))'} = \frac{1+Aw(z)}{1+Bw(z)} = p(z), \quad w \in U \tag{1.2}$$

and that $f \in M_c(\alpha,A,B)$ if and only if

$$\frac{2zf'(z)+2\alpha z^2 f''(z)}{(1-\alpha)(f(z)+\overline{f(\overline{z})})+\alpha z(f(z)+\overline{f(\overline{z})})'} = \frac{1+Aw(z)}{1+Bw(z)} = p(z), \quad w \in U \tag{1.3}$$

where

$$p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \tag{1.4}$$

We study the classes $M_s(\alpha,A,B)$ and $M_c(\alpha,A,B)$, the coefficient estimates for functions belonging to these classes are obtained.

2. Preliminary result

We need the following lemma for proving our results.

Lemma 2.1.([3]) *If $p(z)$ is given by (1.4) then*

$$|p_n| \le A-B, \quad n = 1,2,3,\dots \tag{2.1}$$

3. Main result

We give the coefficient inequalities for the classes $M_s(\alpha,A,B)$ and $M_c(\alpha,A,B)$.

Theorem 3.1. *Let $f \in M_s(\alpha,A,B)$. Then for $n \ge 1$, $0 \le \alpha \le 1$,*

$$|a_{2n}| \le \frac{A-B}{2^n n!(1+(2n-1)\alpha)} \prod_{j=1}^{n-1}(A-B+2j), \tag{3.1}$$

$$|a_{2n+1}| \le \frac{A-B}{2^n n!(1+2n\alpha)} \prod_{j=1}^{n-1}(A-B+2j). \tag{3.2}$$

Proof. From (1.2) and (1.4), we have

$$\begin{aligned}&(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots)\\ &\quad +\alpha(2a_2z^2+6a_3z^3+12a_4z^4+20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots)\\ &= \big[(1-\alpha)(z+a_3z^3+a_5z^5+\cdots+a_{2n-1}z^{2n-1}+a_{2n+1}z^{2n+1}+\cdots)\end{aligned}$$

$$+\alpha(z+3a_3z^3+5a_5z^5+\cdots+(2n-1)a_{2n-1}z^{2n-1}+(2n+1)a_{2n+1}z^{2n+1}+\cdots)]$$
$$\cdot(1+p_1z+p_2z^2+p_3z^3+p_4z^4+\cdots+p_{2n-1}z^{2n-1}+p_{2n}z^{2n}+\cdots)$$

Equating the coefficients of like powers of z, we have

$$2(1+\alpha)a_2 = p_1,\ \ 2(1+2\alpha)a_3 = p_2 \tag{3.3}$$

$$\begin{aligned} 4(1+3\alpha)a_4 &= p_3+(1+2\alpha)a_3p_1 \\ 4(1+4\alpha)a_5 &= p_4+(1+2\alpha)a_3p_2 \end{aligned} \tag{3.4}$$

$$2n(1+(2n-1)\alpha)a_{2n} = p_{2n-1}+(1+2\alpha)a_3p_{2n-3}+\cdots+(1+(2n-2)\alpha)a_{2n-1}p_1 \tag{3.5}$$

$$(2n+1)(1+2n\alpha)a_{2n+1} = p_{2n}+(1+2\alpha)a_3p_{2n-2}+\cdots+(1+(2n-2)\alpha)a_{2n-1}p_2 \tag{3.6}$$

Using lemma 2.1 and (3.3), we get

$$|a_2| \le \frac{A-B}{2(1+\alpha)}, \quad |a_3| \le \frac{A-B}{2(1+2\alpha)}. \tag{3.7}$$

Again by applying (3.6) and followed by Lemma 2.1, we get from (3.4)

$$|a_4| \le \frac{(A-B)(A-B+2)}{(2)(4)(1+3\alpha)}, \quad |a_5| \le \frac{(A-B)(A-B+2)}{(2)(4)(1+4\alpha)}.$$

It follows that (3.1) and (3.2) hold for $n=1,2$. We prove (3.1) using induction.

Equation (3.5) in conjuction with lemma 2.1 yield

$$|a_{2n}| \le \frac{A-B}{2n(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}(1+2k\alpha)|a_{2k+1}|\right]. \tag{3.8}$$

We assume that (3.1) holds for $k=3,4,\ldots,(n-1)$. Then from (3.8), we obtain

$$|a_{2n}| \le \frac{A-B}{2n(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right]. \tag{3.9}$$

In order to complete the proof, it is sufficient to show that

$$\begin{aligned} &\frac{A-B}{2m(1+(2m-1)\alpha)}\left[1+\sum_{k=1}^{m-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right] \\ &= \frac{A-B}{2^m m!(1+(2m-1)\alpha)}\prod_{j=1}^{k-1}(A-B+2j), \quad (m=3,4,5,\ldots,n). \end{aligned} \tag{3.10}$$

(3.10) is valid for $m=3$.

Let us suppose that (3.10) is true for all m, $3<m\le(n-1)$. Then from (3.9)

$$\frac{(A-B)}{2n(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$
\begin{aligned}
&= \left(\frac{n-1}{n}\right)\left(\frac{A-B}{2(n-1)(1+(2n-1)\alpha)}\left(1+\sum_{k=1}^{n-2}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right)\right)\\
&\quad + \frac{(A-B)}{(2n)(1+(2n-1)\alpha)}\,\frac{(A-B)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\\
&= \left(\frac{n-1}{n}\right)\frac{(A-B)}{2^{n-1}(n-1)!(1+(2n-1)\alpha)}\prod_{j=1}^{n-2}(A-B+2j)\\
&\quad + \frac{(A-B)}{2n(1+(2n-1)\alpha)}\;\frac{(A-B)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\\
&= \frac{(A-B)}{2n(n-1)!2^{n-1}(1+(2n-1)\alpha)}\prod_{j=1}^{n-2}(A-B+2j)(A-B+2(n-1))\\
&= \frac{(A-B)}{2^n n!(1+(2n-1)\alpha)}\prod_{j=1}^{n-1}(A-B+2j).
\end{aligned}
$$

Thus (3.10) holds for $m=n$ and hence (3.1) follows. Similarly we can prove (3.2). □

Theorem 3.2. *Let $f \in M_c(\alpha, A, B)$. Then for $n \geq 1$, $0 \leq \alpha \leq 1$,*

$$|a_{2n}| \leq \frac{(A-B)}{(2n-1)!(1+(2n-1)\alpha)}\prod_{j=1}^{2n-2}(A-B+j), \tag{3.11}$$

$$|a_{2n+1}| \leq \frac{(A-B)}{(2n)!(1+2n\alpha)}\prod_{j=1}^{2n-1}(A-B+j). \tag{3.12}$$

Proof. From (1.3) and (1.4), we have

$$
\begin{aligned}
&(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots)\\
&\quad+\alpha(2a_2z^2+6a_3z^3+12a_4z^4+20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots)\\
&=\big[(1-\alpha)(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots+a_{2n}z^{2n}+\cdots)\\
&\quad+\alpha(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots)\big]\\
&\quad\cdot(1+p_1z+p_2z^2+p_3z^3+p_4z^4+\cdots+p_{2n-1}z^{2n-1}+\cdots)
\end{aligned}
$$

Equating the coefficients of like powers of z, we have

$$(1+\alpha)a_2 = p_1,\;\; 2(1+2\alpha)a_3 = p_2+(1+\alpha)a_2p_1, \tag{3.13}$$

$$3(1+3\alpha)a_4 = p_3+(1+\alpha)a_2p_2+(1+2\alpha)a_3p_1, \tag{3.14}$$

$$4(1+4\alpha)a_5 = p_4+(1+\alpha)a_2p_3+(1+2\alpha)a_3p_2+(1+3\alpha)a_4p_1 \tag{3.15}$$

$$(2n-1)(1+(2n-1)\alpha)a_{2n} = p_{2n-1}+(1+\alpha)a_2p_{2n-2}+\cdots+(1+(2n-2)\alpha)a_{2n-1}p_1 \tag{3.16}$$

$$2n(1+2n\alpha)a_{2n+1} = p_{2n}+(1+\alpha)a_2p_{2n-1}+\cdots+(1+(2n-2)\alpha)a_{2n}p_1 \tag{3.17}$$

By using lemma 2.1 and (3.13), we get

$$|a_2| \le \frac{(A-B)}{1+\alpha}, \quad |a_3| \le \frac{(A-B)(A-B+1)}{2(1+2\alpha)}. \tag{3.18}$$

Again by applying (3.18) and followed by Lemma 2.1, we get from (3.14) and (3.15), we have

$$|a_4| \le \frac{(A-B)(A-B+1)(A-B+2)}{(2)(3)(1+3\alpha)},$$
$$|a_5| \le \frac{(A-B)^2+6(A-B)^3+11(A-B)^2+6(A-B)}{(2)(3)(4)(1+4\alpha)}.$$

It follows that (3.11) hold for $n=1,2$. We now prove (3.11) using induction.

Equation (3.16) in conjuction with lemma 2.1 yield

$$|a_{2n}| \le \frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}|a_{2k}|+\sum_{k=1}^{n-1}|a_{2k+1}|\right]. \tag{3.19}$$

We assume that (3.11) holds for $k=3,4,\ldots,(n-1)$. Then from (3.19), we obtain

$$|a_{2n}| \le \frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]. \tag{3.20}$$

In order to complete the proof, it is sufficient to show that

$$\frac{(A-B)}{(2m-1)(1+(2m-1)\alpha)}\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{2(k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]$$
$$=\frac{(A-B)}{(2m-1)!(1+(2m-1)\alpha)}\prod_{j=1}^{2m-2}(A-B+j), \quad (m=3,4,5,\ldots,n). \tag{3.21}$$

(3.21) is valid for $m=3$.

Let us suppose that (3.21) is true for all m, $3<m\le(n-1)$. Then from (3.20)

$$\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]$$
$$=\frac{(2n-3)}{(2n-1)}\left(\frac{(A-B)}{(2(n-1)-1)(1+(2n-1)\alpha)}\left(1+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right)\right)$$
$$+\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{(2(n-1)!-1)}\prod_{j=1}^{2n-4}(A-B+j)$$

$$
\begin{aligned}
&+\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{2(n-1)!}\prod_{j=1}^{2n-3}(A-B+j)\\
&=\frac{2n-3}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\\
&\quad+\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\\
&\quad+\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{2(n-1)!}\prod_{j=1}^{2n-3}(A-B+j)\\
&=\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)(A-B+2n-3)\\
&\quad+\frac{(A-B)}{(2n-1)(1+(2n-1)\alpha)}\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\\
&=\frac{(A-B)}{(2n-1)!(1+(2n-1)\alpha)}\prod_{j=1}^{2n-2}(A-B+j).
\end{aligned}
$$

Thus (3.21) holds for $m=n$ and hence (3.11) follows. Similarly we can prove (3.12). □

On specializing the values of α in Theorem 3.1 and Theorem 3.2, we get the following.

Remark 3.1. In Theorem 3.1, if we set $\alpha=0$, we get starlike functions with respect to symmetric points and if we set $\alpha=1$, we get convex functions with respect to symmetric points.

Remark 3.2. In Theorem 3.2, if we set $\alpha=0$, we get starlike functions with respect to conjugate points and if we set $\alpha=1$, we get convex functions with respect to conjugate points. For other values of α the transition is smooth.

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