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JUST EXCELLENCE AND VERY EXCELLENCE IN GRAPHS WITH RESPECT TO STRONG DOMINATION

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Abstract. A graph G is said to be excellent with respect to strong domination if each $u \in V(G)$, belongs to some γ_s-set of G. G is said to be just excellent with respect to strong domination if each $u \in V(G)$ is contained in a unique γ_s-set of G. A graph G which is excellent with respect to strong domination is said to be very excellent with respect to strong domination if there is a γ_s-set D of G such that to each vertex $u \in V - D$, there exists a vertex $v \in D$ such that $(D - \{v\}) \cup \{u\}$ is a γ_s-set of G. In this paper we study these two classes of graphs. A strong very excellent graph is said to be rigid very excellent with respect to strong domination if the following condition is satisfied. Let D be a very excellent γ_s-set of G. To each $u \notin D$, let $E(u, D) = \{v \in D : (D - \{v\}) \cup \{u\}$ is a γ_s-set of $G\}$. If $|E(u, D)| = 1$ for all $u \notin D$ then D is said to be a rigid very excellent γ_s-set of G. If G has at least one rigid very excellent γ_s-set of G then G is said to be a rigid very excellent graph with respect to strong domination (or) a strong rigid very excellent graph. Some results regarding strong very excellent graphs are obtained.

Introduction

Prof. N. Sridharan and M. Yamuna have introduced the concepts of just excellence and very excellence in graphs. A graph G is said to be excellent if given any vertex u, there is a γ-set of G containing u. A graph G is said to be just excellent if for each vertex $u \in V$, there is a unique γ-set of G containing u. A graph G is very excellent if G is excellent and if there is a γ-set S of G such that to each vertex $u \in V - S$, there exists a vertex $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is a γ-set of G. A γ-set S of G satisfying this property is called *a very excellent γ-set of G*.

Prof. E. Sampathkumar and Pushpalatha have introduced the concept of Strong (weak) domination. A subset D of $V(G)$ is called *a strong dominating set* if for every vertex $v \in V - D$, there exists $u \in D$ such that $uv \in E(G)$ and $\deg u \geq \deg v$. A strong dominating set of minimum cardinality is called *a minimum strong dominating set* and its cardinality is called *the strong domination number*. The strong domination number is denoted by γ_s and a minimum strong dominating set is called *a γ_s-set.*

A subset D of $V(G)$ is called *a weak dominating set of* G if for every vertex $v \in V - D$, there exists $u \in D$ such that $uv \in E(G)$ and $\deg u \leq \deg v$. A weak dominating set of minimum cardinality is called *a minimum weak dominating set* and its cardinality is

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called *the weak domination number*. The weak domination number is denoted by γ_w and a minimum weak dominating set is called a γ_w-set.

Definition 1. A graph G is said to be strong excellent γ_s-excellent, if for a given vertex u of G there exists a γ_s-set of G containing u.

Definition 2. A graph G is said to be strong just excellent (or) shortly γ_s-just excellent if for every $u \in V$, there is a unique γ_s-set containing u.

Definition 3. A graph G is said to be strong very excellent (or) shortly γ_s-very excellent if there is a γ_s-set S of G such that to each vertex $u \notin S$, there exists $v \in S$ with $(S - \{v\}) \cup \{u\}$ a γ_s-set of G. A γ_s-set S of G satisfying this property is called *a very excellent γ_s-set of G*.

Definition 4. A graph G is said to be strong rigid very excellet (or) shortly γ_s-rigid very excellent, if G is strong very excellent and for any very excellent γ_s-set D of G and for any $u \notin D$ there exists a unique $v \in D$ such that $(D - \{v\}) \cup \{u\}$ is a γ_s-set.

Definition 5. Let u and v belong to $V(G)$. Then $\deg_s(u) = |N_s(u)|$ where $N_s(u) = \{v \in V : uv \in E(G),\ \deg v \geq \deg u\}$. Similarly $N_w(u)$ is defined as $N_w(u) = \{v \in V : uv \in E(G),\ \deg v \leq \deg u\}$. $d_w(u)$ is defined as $d_w(u) = |N_w(u)|$. u is said to be a strong isolate if $N_s(u) = \phi$. Similarly a weak isolate can be defined.

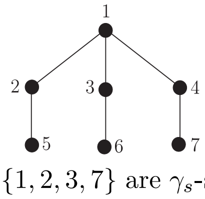
Definition 6. $\delta_s(G) = \min\limits_{u \in V}(d_s(u))$, $\Delta_s(G) = \max\limits_{u \in V}(d_s(u))$, $\delta_w(G) = \min\limits_{u \in V}(d_w(u))$ and $\Delta_w(G) = \max\limits_{u \in V}(d_w(u))$.

Definition 7. If D is a γ_s-set of G, then $PN_w[u, D] = \{v \in V(G) : v$ is strongly dominated by u and v is not strong dominated by $D - \{u\}\} = N_w[u] - N_w[D - \{u\}]$. $PN_w(u, D)$ is defined as $N_w(u) - N_w[D - \{u\}]$. Note that $u \in PN_w[u, D]$ and $u \notin PN_w(u, D)$.

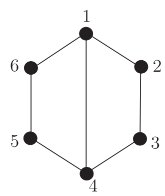
Example 1. Any γ-excellent regular graph is γ_s-excellent.

Example 2. Any double star $K_{r,r}$ is γ_s-rigid excellent but not γ-rigid excellent.

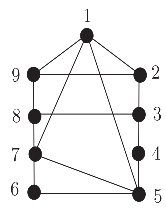
Example 3. K_n is γ_s-very excellent.

Example 4. $G =$ 

is γ_s-very excellent, since $\{1, 2, 3, 4\}$ is a γ_s-set. $\{1, 3, 4, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 2, 3, 7\}$ are γ_s-sets. G is not γ-very excellent.

Example 5. $G =$ 

This is γ-just excellent but not γ_s-just excellent.

Example 6. $G =$  is γ_s-just excellent, since $\{1, 3, 6\}$, $\{5, 8, 9\}$ and $\{2, 4, 7\}$ are γ_s-sets.

1. Just Excellent Graphs

Observation 1. *If G is γ_s-just excellent and $G \neq K_n$ then $N_w[u] \neq N_w[v]$ for any $u, v \in V(G)$, where $\{N_w[u] = \{\{u\} \cup \{v \in V : uv \in E(G);\ d(u) \geq d(v)\}\}$.*

Proof. Since G is γ_s-just excellent, there exists a unique γ_s-set say D containing u. Suppose there exists a $v \in V(G)$ such that $N_w[u] = N_w[v]$. Then $(S - \{u\}) \cup \{u\}$ is a γ_s-set. Since $G \neq K_n$ and since any non-regular graph with a vertex of degree $(n-1)$ is not γ_s-rigid excellent, $|S| > 1$. Therefore, every vertex of $S - \{u\}$ lies in at least two γ_s-excellent sets namely S and $(S - \{u\}) \cup \{v\}$ contradicting the γ_s-rigid excellence of G. Hence the observation.

Observation 2. *If G is γ_s-excellent then $\delta_s(u) \geq \frac{n}{\gamma_s(G)} - 1$.*

Proof. Let us assume that $V = S_1 \cup S_2 \cup \cdots \cup S_m$ where each S_i is a γ_s-set. Let $m \geq 2$. Let $u \in S_j$. Since each S_i is a γ_s-set, u is strongly dominated by some point $v \in S_i$, $i \neq j$. Hence $\delta_s(u) \geq m - 1 = \frac{n}{\gamma S(G)} - 1$. If $m = 1$ the V is a γ_s-set. Hence G is totally disconnected. Therefore, $\delta_s(u) = 0 = \frac{n}{n} - 1 = \frac{n}{\gamma_s(G)} - 1$.

Observation 3. *If $G \neq K_2$ and $G \neq \overline{K_n}$ and if G is γ_s-just excellent then $\delta_s(u) \geq 2$ (In particular any tree $\neq K_2$ is not γ_s-just excellent).*

Proof. Let $G \neq K_2$, $G \neq \overline{K_n}$. Let $\delta_s(u) = 1$ for some $u \in V(G)$. Let $N_s(u) = \{v\}$. Since G is just excellent there exists a γ_s-set D of G, containing u. If $v \in D$, then there are two γ_s-sets containing v, since D and $(D - \{u\}) \cup \{v\}$ are two γ_s-sets containing v. Therefore, $v \notin D$, since $G \neq K_2$, $|D| \geq 2$. Therefore, $(D - \{u\}) \cup \{v\}$ is a γ_s-set or G and hence every element of $D - \{u\}$ is contained in at least two γ_s-sets, a contradiction.

Lemma 1. *Every γ_s-just excellent graph $G \neq \overline{K_n}$ is connected.*

Proof. If G is not connected, by hypothesis, one of the connected components say G_1 of G contains more than one vertex. Since G is γ_s-just excellent, G_1 is γ_s-just excellent and G_1 is connected, $\gamma_s(G_1) < |V(G_1)|$. Since G_1 is γ_s-just excellent, G_1 has at least two γ_s-sets. Let S_1, S_2 be two γ_s-sets of G_1. Let D be a γ_s-set of $G - G_1$. Then both $D \cup S_1$ and $D \cup S_2$ are γ_s-set of G containing D, a contradiction to the fact that G is γ_s-just excellent. Hence G is connected.

Lemma 2. *If G is strong just excellent and $G \neq \overline{K_n}$, then G has no strong isolates.*

Proof. Since G is strong just excellent and $G \neq \overline{K_n}$, by the above lemma, G is connected. Then $\gamma_s(G) < |V(G)|$. Therefore G has at least two γ_s-sets. If G has a strong isolate, then this belongs to every γ_s-set, a contradiction. Hence G has no strong isolates.

Definition 8. Let D be a subset of V. Then $\langle D \rangle$, called *the induced subgraph of* G, is defined as the subgraph with vertex set D and two vertices in this subgraph are adjacent if they are adjacent in G.

Lemma 3. *If $G \neq K_n$ and G is γ_s-just excellent then $|PN_w(u, D)| \geq 2$ for all $u \in D$, where D is a γ_s-set of G, and u is not a strong isolate of $\langle D \rangle$.*

Proof. Let D be a γ_s-set of G. If $PN_w(u, D) = \phi$, then $(D - \{u\}) \cup \{w\}$ is also a γ_s-set of G, for any w in $N_s(u)$. (Note that $N_s(u) \neq \phi$ as G has no strong isolates). If $D = \{u\}$ then, $G = K_n$ a contradiction. Therefore, $D - \{u\}$ contains a point and hence every point in $D - \{u\}$ is contained in at least two γ_s-sets, namely D and $(D - \{u\}) \cup \{w\}$, a contradiction since G is strong just excellent. Suppose $|PN_w(u, D)| = 1$. Let $PN_w(u, D) = \{w\}$.

Then $(D - \{u\}) \cup \{w\}$ is a γ_s-set (since u is not a strong isolate of $\langle D \rangle$). Noting that D has at least two points we get that every vertex in $(D - \{u\})$ is in at least two γ_s-sets, namely D and $(D - \{u\}) \cup \{w\}$, a contradiction. Hence the theorem.

Remark 1. If G is γ_s-just excellent and if S is a γ_s-set of G, then a vertex in $V - S$ may be strong dominated by more than one vertex of S. For example, in Example 6, the vertex 2 is strong dominated by two vertices of the γ_s-set $\{1, 3, 6\}$.

Theorem 1. *Let $G \neq \overline{K_n}$ be just excellent. Let $\gamma_s(G) = k$. Then $\Delta_w(G) \leq n - k$.*

Proof. Let $u \in V(G)$. Let S be a γ_s-set of G which contains u. $|PN_w(V - S)| \geq 1$ for all $v \in S$. Therefore, u is not strong adjacent to any point in $\bigcup_{v \neq u, v \in S} PN_w(v, S)$. Therefore, $d_w(u) \leq (n - 1) - (k - 1) = n - k$. Therefore, $\Delta_w(G) \leq n - k$.

Definition 9. The strong domatic number of G, denoted by $d_s(G)$ is defined as the maximum cardinality of partition of V into strong dominating sets of G. Note that since V is a strong dominating set $d_s(G) \geq 1$.

Lemma 4. *The graph G is just excellent if and only if all of the following conditions hold.*

1. *$\gamma_s(G)$ divides n.*
2. *G has exactly $\frac{n}{\gamma_s(G)}$ distinct γ_s-sets.*
3. *$d_s(G) = \frac{n}{\gamma_s(G)}$.*

Proof. Let G be just excellent. Let $S_1, S_2, S_3, \ldots, S_m$ be the collection of distinct γ_s-sets of G. Then $S_1, S_2, S_3, \ldots, S_m$ is a partition of V into m γ_s-sets. Therefore

$m\gamma_s(G) = n$. Therefore (1) and (2) follows. Since $S_1, S_2, S_3, \ldots, S_m$ provide a domatic partition with $m = \frac{n}{\gamma_s(G)}$ we get that $d_s(G) = \frac{n}{\gamma_s(G)}$.

Conversely assume that G satisfies conditions 1-3. Then $m\gamma_s(G) = n$. Since $d_s(G) = \frac{n}{\gamma_s(G)} = m$, there exists a decomposition of $V(G)$ into m strong dominating sets of G, say $S_1, S_2, S_3, \ldots, S_m$. Then $|S_i| \geq \gamma_s$. Therefore $n = \sum_{i=1}^{m} |S_i| \geq m\gamma_s$. Therefore $m\gamma_s(G) = \sum_{m}^{i=1} |S_i| \geq m\gamma_s$. Therefore each S_i is a γ_s-set. By hypothesis G has exactly $\frac{n}{\gamma_s(G)}$ distinct γ_s-sets. That is G has exactly m distinct γ_s-sets. Therefore, $S_1, S_2, S_3, \ldots, S_m$ are precisely m distinct γ_s-sets. Also each vertex belongs to exactly one S_i (since $\{S_1, S_2, S_3, \ldots, S_m\}$ is a partition of V). Therefore, G is γ_s-just excellent.

Theorem 2. *Let $u \in V$. Let D be the unique γ_s-set containing u. Let t be the number of strong isloates of $\langle D\rangle$.*

Then $d_w(u) \leq \begin{cases} n - 2\gamma_s + 2t - 1 & \text{if } u \text{ is not a strong isolate of } D \\ n - 2\gamma_s + 2t - 3 & \text{if } u \text{ is a strong isolate of } D \end{cases}$

Proof. For any non-strong isolate v of D, $|PN_w(v, D)| \geq 2$. Also, if $v \in D$ and if $x \in PN_w(v, D)$ then u does not strong dominate x.

$$\text{Therefore } d_w(u) \leq \begin{cases} (n-1) - 2(\gamma_s - 1 - t) + t & \text{if } u \text{ is not a strong isolate,} \\ (n-1) - 2[\gamma_s - 1 - (t-1)] + (t-1) & \text{if } u \text{ is a strong isolate.} \end{cases}$$

$$\text{Therefore } d_w(u) \leq \begin{cases} n - 2\gamma_s + 3t + 1 & \text{if } u \text{ is not a strong isolate,} \\ n - 2\gamma_s + 3t - 2 & \text{if } u \text{ is a strong isolate.} \end{cases}$$

Corollary 1. *If D has no strong isolates then $d_w(u) \leq n - 2\gamma_s + 1$.*

Theorem 3. *Let $G \neq K_n$ be just excellent. Then $\gamma_s(G) \leq \frac{n}{3}$.*

Proof. Suppose $d_s(G) = 2$. Then $V = S_1 \cup S_2$ where S_1 and S_2 are distince γ_s-sets of G. For any $u \in S_1$, $PN_w(u, S_1) \subseteq S_2$. Suppose for some $u \in S_1$, $|PN_w(u, S_1)| \geq 2$. Then $|S_2| \geq |S_1| + 1$. But $2\gamma_s = |S_1| + |S_2| \geq 2|S_1| + 1 \geq 2\gamma_s + 1$, a contradiction. Therefore, every point of S_1 is a strong isolate of S_1. Similarly every point of S_2 is a strong isolate of S_2. Suppose $|PN_w(u, S_1)| \geq 2$ for some point $u \in S_1$, by the above argument we get that $2\gamma_s \geq \gamma_s + 1$, a contradiction. Therefore, $|PN_w(u, S_1)| = 1$ for $u \in S_1$. Similarly this result is true for S_2 also.

Let $S_1 = \{u_1, u_2, u_3, \ldots, u_k\}$, $S_2 = \{v_1, v_2, v_3, \ldots, v_k\}$. Without loss of generality let $\{v_i\} = PN_w(u_i, S_1)$. Then $d(u_i) > d(v_i)$. If $d(u_i) = d(v_i)$ then $(S_1 - \{u_i\}) \cup \{v_i\}$ is a γ_s-set and so, every point of $(S_1 - \{u_i\})$ lies in two γ_s-sets namely S_1 and $(S_1 - \{u_i\}) \cup \{v_i\}$, a contradiction.

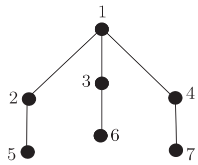
Suppose u_i and u_j are adjacent, as u_i and u_j are strong isolates, $d(u_i) > d(u_j)$ and $d(u_j) > d(u_i)$, a contradiction. Therefore, u_i and u_j are not adjacent. That is $\langle S_1\rangle$ is totally disconnected. The same is true for S_2 also. Let $u_1, u_2, \ldots, u_k$ be such that $d(u_1) \leq d(u_2) \leq \cdots \leq d(u_k)$. We have $d(u_1) > d(v_1)$. Also $d(v_1) > d(u_s)$ for some $s > 1$ (where $\{u_s\} = PH_w(v_1, S_2)$). Therefore, $d(u_1) < d(u_s) < d(v_1) < d(u_1)$, a contradiction. Therefore $d_s(G) \geq 3$. Since $n = \gamma_s(G)d_s(G)$, we get that $\gamma_s(G) = \frac{n}{d_s(G)} \leq \frac{n}{3}$.

Remark 2. For C_{3n}, $\gamma_s(C_{3n}) = n$ and C_{3n} is γ_s-just excellent.

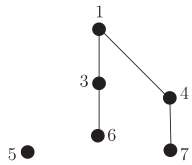
Definition 10. Let $u \in V(G)$. A subset S of minimum cardinality such that S strong dominates $G - \{u\}$ is called *a* $\gamma_s^u(G,u)$ *set of* G.

Definition 11. $u \in V$ is said to be a γ_s level vertex of G, if $\gamma_s^u(G,u) = \gamma_s(G)$. u is said to be a γ_s-non-level vertex of G, if $\gamma_s^u(G,u) = \gamma_s(G) - 1$.

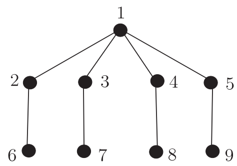
Example 7. G :



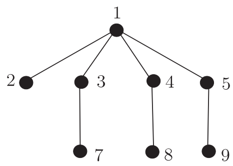
$G - 2$:



Example 8. G :



$G - 6$:



In Example 7, $\{2,3,4\}$ and $\{3,4,5\}$ are subsets of V of minimum cardinality which dominate $G - \{2\}$. Therefore $\gamma_s^2(G,2) = 3$. 2 is a γ_s-non level vertex of G. In Example 8 $\gamma_s(G) = \gamma_s^8(G,8) = 5$. Therefore 8 is a γ_s-level vertex.

Theorem 4. *Let G be a γ_s-just excellent graph, $G \neq \overline{K_n}$. Then every vertex u is a γ_s-level vertex and $\gamma_s(G - \{u\}) = \gamma_s(G)$.*

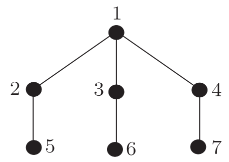
Proof. If $G = K_n$, the theorem is obviously true. Let $G \neq K_n$ and $G \neq \overline{K_n}$. Let u be a vertex in G. Since G is γ_s-just excellent, there exists a γ_s-set S of G not containing u. Clearly S strong dominates $G - \{u\}$. Therefore, $\gamma_s(G - \{u\}) \leq |S| \leq \gamma_s(G)$. Suppose $\gamma_s(G - \{u\}) < \gamma_s(G)$. Let T be a γ_s-set of $G - \{u\}$. Then $T \cup \{v\}$ is a γ_s-set for G, for every $v \in N_s[u]$. $N_s[u]$ contains at least two points, since u is not a strong isolate. Therefore, there exists a point in $N_s[u]$ different from u which strong dominates u. Let $v \in N_s(u)$. Then $T \cup \{v\}$ and $T \cup \{u\}$ are γ_s-sets containing T. Therefore, every element of T is contained in at least two r_s-sets of G, a contradiction to γ_s-just excellence of G. Therefore, $\gamma_s(G - \{u\}) = \gamma_s(G)$.

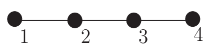
Suppose $\gamma_s^u(G,u) < \gamma_s(G)$. Let $S \subseteq V$ be a $\gamma_s^u(G,u)$-set of G. If $u \in S$, then S is γ_s-set of G, a contradiction. Therefore, $u \notin S$. Therefore S is a strong dominating set for $G - u$. Therefore, $\gamma_s(G - \{u\}) \leq |S| < \gamma_s(G)$, a contradiction. Since $\gamma_s(G - \{u\}) = \gamma_s(G)$, $\gamma_s^u(G, \{u\}) = \gamma_s(G)$. Hence the theorem.

2. Strong Very Just Excellent Graphs

We recall the definition of strong very excellent (or) γ_s-very excellent graphs.

A strong excellent graph G is said to be strong very excellent if there is a γ_s-set S of G such that to each vertex $u \in V - S$, there exists a vertex v in S such that $(S - \{v\}) \cup \{u\}$ is a γ_s-set of G. A γ_s-set of G satisfying the above property is called *very just excellent γ_s-set of G.*





In first example, the graph is strong very just excellent and $\{1, 2, 3, 4\}$ is a very just excellent γ_s-set. In second example $\{1, 3\}$, $\{2, 3\}$, $\{2, 4\}$ is very excellent and $\{2, 3\}$ is very just excellent γ_s-set.

Theorem 5. *P_n is γ_s-very excellent if and only if $n = 2$ or $n = 4$.*

Proof. It has already been proved in [2] that P_n is γ_s-excellent if and only if $n = 2$ or $n \equiv 1 \pmod 3$. P_2, P_4 are obviously γ_s-very excellent. Consider a path P_n : $v_1, v_2, v_3, \ldots, v_n$ where $n = 3k+1$, $k \geq 2$. Let S be any γ_s-set for P_n. Then at least $\gamma_s - 2$ vertices are isolated in $\langle S \rangle$. To each $u \in S$, let $PN_w(u) = \{v \in V(P_n) : N_s(v) \cap S = \{u\}\}$. Suppose $|PN_w(u)| = 2$ for some $u \in S$. Let v_i be the point in S such that $|PN_w(v_i)| = 2$, $2 \leq i \leq 3k$ (Note that $d(u) \leq 2$ for all $u \in V(P_n)$).

Subcase(1):

Suppose $i = 2$. Since v_1 is strongly dominated only by v_2 in S, $S - \{v_2\} \cup \{v_j\}$, $j \neq 1$ is not a strong dominating set. Also $(S - \{v_2\}) \cup \{v_1\}$ is also not a strong dominating set. (For, since $v_1, v_3 \in PN_w(v_2)$. v_3 is a weak private neighbour of v_2. Since v_2 is dropped the newly introduced point v_1 must dominate v_3 strongly which is not true since v_1 is not even adjacent to v_3). This contradicts the fact that S is a strong very excellent γ_s-set. By similar reasoning would prove for $i \neq 3k$.

Subcase(2):

Let $2 < i < 3k$. It is enough, if we prove for $2 < i < \frac{3k+1}{2}$ (a similar reasoning would prove for $3k > 1 > \frac{3k+1}{2}$). As $|PN_w(v_i)| = 2$, $v_{i-1}, v_{i+1} \in PN_w(v_i)$. $v_{i+2} \notin S$ (for if $v_{i+2} \in S$, then v_{i+1} is not in the weak private neighbour of v_{i+2}, a contradiction.) Clearly $v_{i+3} \in S$. If v_{i+3} is an isolate in $\langle S \rangle$, then $(S - \{v_{i+3}\}) \cup \{v_{i+1}\}$ does not strong dominate v_{i+3}. Then for inclusion of v_{i+1} in S, there exists no point in S whose deletion will result in a γ_s-set. Therefore, v_{i+3} is not an isolate in $\langle S \rangle$. Therefore, $v_{i+4} \in S$. Since $n = 3k + 1$, $i + 3$ and $i + 4$ cannot be the last two points of the path. Therefore, $i + 4 \leq n - 1$ and $v_{i+5} \in PN_w(v_{i+4})$. Let Q_1 denote $v_1, v_2, \ldots, v_{i-2}$ path and Q_2 denote $v_{i+6}, v_{i+7}, \ldots, v_n$. (Any one of the paths Q_1 or Q_2 may be empty). Then

$(S-\{v_i, v_{i+3}, v_{i+4}\})$ dominates the vertices of Q_1 and Q_2. As $n \geq 10$, $Q_1 \neq \phi$ or $Q_2 \neq \phi$. $|Q_1 \cup Q_2| = n - 7 = 3k + 1 - 7 = 3k - 6 \equiv 0 (\bmod 3)$.

$|S - \{v_i, v_{i+3}, v_{i+4}\}| = k - 2$. And so no vertex in Q_1 is adjacent to any vertex in Q_2, $\{v_i, v_{i+3}, v_{i+4}\}$ does not dominate any vertex in $Q_1 \cup Q_2$. The set $S \cap Q_1$ strong dominates Q_1 and $S \cap Q_2$ strong dominates Q_2. So

$$|S \cap Q_1| = \left\lceil \frac{|Q_1|}{3} \right\rceil; \quad |S \cap Q_2| = \left\lceil \frac{|Q_2|}{3} \right\rceil.$$
$$|S \cap (Q_1 \cup Q_2)| = k - 2 = \frac{3k+1-7}{3} = \left| \frac{Q_1 \cup Q_2}{3} \right|.$$

We have $|S \cap Q_1| + |S \cap Q_2| = |S \cap \{(Q_1 \cup Q_2)\}| = \frac{|Q_1 \cup Q_2|}{3}$. That is, $|S \cap Q_1| + |S \cap Q_2| = \frac{|Q_1 \cup Q_2|}{3}$. That is, $\left\lceil \frac{|Q_1|}{3} \right\rceil + \left\lceil \frac{|Q_2|}{3} \right\rceil = \frac{|Q_1 \cup Q_2|}{3}$. Suppose $|Q_1|$ and $|Q_2|$ are not both divisible by 3. Let $|Q_1| = 3l+1$ or $3l+2$. Let $|Q_2| = 3m+2$ or $3m+1$ (note that since $|Q_1|+|Q_2|$ is divisible by 3, $|Q_1| = 3l+1$ and $|Q_2| = 3m+2$ or $|Q_1| = 3l+2$ and $|Q_2| = 3m+1$). Therefore, $\left\lceil \frac{|Q_1|}{3} \right\rceil + \left\lceil \frac{|Q_2|}{3} \right\rceil = l+1+m+1 = l+m+2$. $\frac{|Q_1 \cup Q_2|}{3} = \frac{3l+1+3m+2}{3} = l+m+1$, a contradiction.

Suppose $|Q_1|$ is not divisible by 3 and $|Q_2|$ is divisible by 3. Let $|Q_1| = 3l+1$ or $3l+2$ and $|Q_2| = 3m$. $\left\lceil \frac{|Q_1|}{3} \right\rceil + \left\lceil \frac{|Q_2|}{3} \right\rceil = l+1+m+1$. $\frac{|Q_1 \cup Q_2|}{3} = \frac{3l+1+3m}{3}$ or $\frac{3l+3m+2}{3}$ is not an integer, a contradiction. Similarly $|Q_1|$ is divisible by 3 and $|Q_2|$ is not divisible by 3 is also not true. Therefore, $|Q_1|$ and $|Q_2|$ are divisible by 3. If $Q_1 \neq \phi$ then $v_2 \in S$ and $v_1, v_3 \in PN_w(v_2)$. Then $(S - \{w\}) \cup \{v_1\}$ is not a strong dominating set for any $w \in S$. If $Q_2 \neq \phi$, then as $|Q_2|$ is divisible by 3, we get that $i+4$ or $i+10$ or $\cdots$ or $i+3t+1$, $(t \geq 2)$ will be the last but one point of the path P_n. Therefore, $i+3t+2$ belongs to S. That is, v_{n-1} belongs to S and $v_{n-2}, v_n \in PN_w(v_{n-1})$.

In the case for the inclusion of $v_n \in S$, there exists no point in S whose deletion will result in a γ_s-set, a contradiction. Therefore, if S is a strong very excellent set, then

$$|PN_w(u)| \leq 1 \tag{1}$$

for every $u \in S$. Let $n \geq 13$. Let P' be the $v_1 - v_{10}$ path and P'' be the $v_{11} - v_n$ path. If $S \cap P'$ does not strong dominate any of the vertices of P'', then $S \cap P''$ is a γ_s-set for P''. As $|p''| \equiv 0 (\bmod 3)$ and $\{v_j : j = 3t,\ 4 \leq t \leq k\}$ is the unique γ_s-set for P''. It follows that $v_{n-1} \in S$ and therefore, $|PN_w(v_{n-1})| = 2$, a contradiction to (1). If $S \cap P'$ dominates a vertex of P'', then $v_{10} \in S$. Hence $|S \cap P'| = 4$. If $v_{11} \in PN_w(v_{10})$, then $S \cap P'$ is a γ_s-set for $v_1 - v_{11}$ path. Then $PN_w(u) = 2$ for at least two points in $S \cap P''$, a contradiction to (1). If $v_{11} \notin PN_w(v_{10})$, then $v_{12} \in S$ and $S \cap P''$. Therefore, S contains v_{n-1} and $|PN_w(v_{n-1})| = 2$, a contradiction to (1).

If $n = 10$, then the γ_s-sets are $S_1 = \{v_2, v_5, v_8, v_{10}\}$, $S_2 = \{v_2, v_5, v_8, v_9\}$, $S_3 = \{v_1, v_3, v_6, v_9\}$, $S_4 = \{v_2, v_5, v_7, v_9\}$, $S_5 = \{v_2, v_4, v_7, v_9\}$. In S_1 and S_2, $|PN_w(v_5)| = 2$. In S_3, $|PN_w(v_3)| = |PN_w(v_6)| = 2$. In S_5, the inclusion of v_1 does not result in a γ_s-set for the deletion of any element of S_5. Therefore, P_{10} is not γ_s-very excellent. If $n = 7$, then the γ_s-sets are $S_1 = \{v_2, v_5, v_7\}$, $S_2 = \{v_2, v_5, v_6\}$, $S_3 = \{v_1, v_3, v_6\}$,

$S_4 = \{v_2, v_3, v_6\}$, $S_5 = \{v_2, v_4, v_6\}$. It can be verified that none of these is a γ_s-very excellent set. Therefore, P_7 is not a strong very excellent. Therefore, the only γ_s-very excellent paths are P_2 and P_4.

Theorem 6. *A graph is γ_s-very excellent if and only if there exists a γ_s-set D of G such that to each $u \notin D$ there exist $v \in D$ such that $PN_w(v, D) \subseteq N_w[u]$.*

Proof. Suppose D satisfies this proterty, then clearly D is a very excellent γ_s-set of G. Conversely suppose G is γ_s-very excellent. Let D be a very excellent γ_s-set of G. Let $u \notin D$. Then there exists a $v \in D$ such that $(D - \{v\}) \cup \{u\}$ is a γ_s-set of G. As $(D - \{v\})$ does not strong dominate any vertex of $PN_w[v, D]$, and as $(D - \{v\}) \cup \{u\}$ is a γ_s-set, u strong dominates $PN_w[v, D]$. That is $PN_w[v, D] \subseteq N_w[u]$. Hence the theorem.

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