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FREE AND CYCLIC CANONICAL $(m,n)-$ARY HYPERMODULES

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Abstract. In this paper, the class of free and cyclic canonical $(m,n)-$ary hypermodules over Krasner $(m,n)-$ary hyperrings is defined. Free canonical $(m,n)-$ary hypermodules are a generalization of free canonical hypermodules and a generalization of free modules. Also, several properties are found. In addition, we introduce the concepts of a free basis and a free (m,n)-hypermodules as a free object in the category of (m,n)-hypermodules and prove some results in this respect. Finally, we obtain some results and relations among a finitely generated torsion free and a free (m,n)-hypermodule.

1. Introduction

The notion of an $n-$ary group which is a natural generalization of the notion of a group, was introduced by Dörnte [8]. Since then many papers concerning various n-ary algebra appeared in the literature, (for example see [5, 6, 9, 10, 11]). n-ary hyperstructures, recently introduced by Davvaz and Vougiouklis are a nice generalization of the algebraic hyperstructures [7], which have been studied since 70 years, both on the theoretical point of view and for the richness of their applications, especially to computer sciences, but also to fuzzy set theory, graphs and hypergraphs, geometry, lattice theory and others. n-ary hyperstructures generalization of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties (for example see [14, 15]). In theoretical aspects, Anvariyeh et al. studied several operations on hypermodules rigorously [1, 2, 3]. Also, in [4] a new class of hyperstructures as a generalization of hypermodules (namely $(m,n)-$ary hypermodules) is introduced and several properties and examples are found. On the other hand, we can consider $(m,n)-$ary hypermodules as a good generalization of $(m,n)-$ary modules. In this paper, a new subclass of (m,n)-ary hypermodules is called free canonical (m,n)-ary hypermodules is introduced. Free canonical $(m,n)-$ary hypermodules are a generalization

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of free canonical hypermodules and free modules [13, 17]. In addition, we define the concepts of a free basis and a free (m,n)-hypermodule as a free object in the category of (m,n)-hypermodules and prove some results in this respect.

Let H be a non-empty set and h a mapping $h: H\times H \longrightarrow \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then h is called a *binary hyperoperation* on H. We denoted by H^n the cartesian product $H\times\ldots\times H$, where appears n times and an element of H^n will be denoted by $(x_1,\ldots,x_n)$, such that $x_i\in H$ for any i with $1\le i\le n$. In general, a mapping $h: H^n \longrightarrow \wp^*(H)$ is called an *$n-$ary hyperoperation* and n is called the *arity of hyperoperation*.

Let h be an $n-$ary hyperoperation on H and $A_1,\ldots,A_n$ subsets of H. We define

$$h(A_1,\ldots,A_n)=\bigcup\{h(x_1,\ldots,x_n)|x_i\in A_i, i=1,\ldots,n\}.$$

We shall use the following abbreviated notation: the sequence $x_i,x_{i+1},\ldots,x_j$ will be denoted by x_i^j. Also, for every $a\in H$, we write $h(\underbrace{a,\ldots,a}_{n})=h(\overset{(n)}{a})$ and for $j<i$, x_i^j is the empty set. In this convention

$$h(x_1,\ldots,x_i,y_{i+1},\ldots,y_j,x_{j+1},\ldots,x_n)$$

will be written $h(x_1^i,y_{i+1}^j,x_{j+1}^n)$.

If h is an $n-$ary groupoid and $t=l(n-1)+1$, then the $t-$ary hyperoperation $h_{(l)}$ given by

$$h_{(l)}(x_1^{l(n-1)+1})=\underbrace{h(h(\ldots,h(h}_{l}(x_1^n),x_{n+1}^{2n-1}),\ldots),x_{(l-1)(n-1)+2}^{l(n-1)+1}),$$

will be denoted by $h_{(l)}$.

A non-empty set H with an $n-$ary hyperoperation $h: H^n \longrightarrow \wp^*(H)$ will be called an *$n-$ary hypergroupoid* and will be denoted by (H,h). An $n-$ary hypergroupoid (H,h) will be an *$n-$ary semihypergroup* if and only if the following associative axiom holds:

$$h(x_1^{i-1},h(x_i^{n+i-1}),x_{n+i}^{2n-1})=h(x_1^{j-1},h(x_j^{n+j-1}),x_{n+j}^{2n-1})$$

for every $i,j\in\{1,2,\ldots,n\}$ and $x_1,x_2,\ldots,x_{2n-1}\in H$.

An $n-$ary semihypergroup (H,h), in which the equation $b\in h(a_1^{i-1},x_i,a_{i+1}^n)$ has the solution $x_i\in H$ for every $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n,b\in H$ and $1\le i\le n$, is called an *$n-$ ary hypergroup*.

An $n-$ary hypergroupoid (H,h) is commutative if for all $\sigma\in\mathbb{S}_n$ and for every $a_1^n\in H$ we have

$$h(a_1,\ldots,a_n)=h(a_{\sigma(1)},\ldots,a_{\sigma(n)}).$$

An element $e\in H$ called a scalar neutral element if $x=h(\overset{(i-1)}{e},x,\overset{(n-i)}{e})$, for every $1\le i\le n$ and for every $x\in H$.

According to [12], an $n-$ary polygroup is an $n-$ary hypergroup (P,f) such that the following axioms hold for $1 \leq i, j \leq n$ and $x, x_1^n \in P$:

(1) There exists a unique element $0 \in P$ such that $x = f(\overset{(i-1)}{0}, x, \overset{(n-i)}{0})$.

(2) There exists a unitary operation $-$ on P such that $x \in f(x_1^n)$ implies that,

$$x_i \in f(-x_{i-1}, \ldots, -x_1, x, -x_n, \ldots, -x_{i+1}).$$

It is clear that every $2-$ary polygroup is a polygroup. Every $n-$ary group with a scaler neutral element is an $n-$ary polygroup. Also, Leoreanu- Fotea in [14] define a canonical $n-ary$ hypergroup. A canonical $n-ary$ hypergroup is a commutative $n-$ary polygroup.

An element 0 of an $n-$ ary semihypergroup (H,g) is called zero element if for every $x_2^n \in H$, we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0) = 0.$$

If 0 and $0'$ are two zero elements, then $0 = g(0', \overset{(n-1)}{0}) = 0'$ and so the zero element is unique.

Mirvakili and Davvaz define $(m,n)-$ary hyperrings and obtained several results in this respect. Now, we define Krasner $(m,n)-$ary hyperrings as a subclasses of $(m,n)-$ary hyperrings and as a generalization of Krasner hyperrings.

Definition 1.1.([18]). A Krasner $(m,n)-$ary hyperring is an algebraic hyperstructure (R,f,g) which satisfies the following axioms:

(1) (R,f) is a canonical $m-$ary hypergroup.

(2) (R,g) is an $n-$ary semigroup.

(3) the $n-$ary operation g is distributive with respect to the $m-$ary hyperoperation f, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, $1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

(4) 0 be a zero element (absorbing element) of $n-$ary operation g, i.e., for every $x_2^{n-1} \in R$, we have

$$g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0) = 0.$$

It is clear that every Krasner hyperring is a Krasner $(2,2)-$hyperring. Also, every Krasner $(m,0)-$hyperring is a canonical $m-$ary hypergroup and every Krasner $(0,n)-$hyperring is an $n-$ary semigroup.

Let (R,f,g) be a Krasner $(m,n)-$ary hyperring. If $(R-\{0\},h)$ is an $n-$ary group then (R,f,g) is an $(m,n)-$ary hyperfield.

Example 1.2. Let $(R,+,\cdot)$ be a Krasner hyperring. Let f be an m-ary hyperoperation and g an $n-$ary operation on R as follows:

$$f(x_1^m) = \sum_{i=1}^{m} x_i, \quad \forall x_1^m \in R,$$
$$g(x_1^n) = \prod_{i=1}^{n} x_i, \quad \forall x_1^n \in R,$$

then (R,f,g) is a Krasner (m,n)-hyperring and denoted by $(R,f,g) = der_{(m,n)}(R,+,\cdot)$.

Example 1.3. Let $(R,+,\cdot)$ be a ring and G a normal subgroup of $(R,.)$, i.e., for every $x \in R$, $xG = Gx$. Set $\bar{R} = \{\bar{x} | x \in R\}$ where $\bar{x} = xG$ and define the $m-$ary hyperoperation f and $n-$ary multiplication g as follows:

$$f(\bar{x}_1,\ldots,\bar{x}_m) = \{\bar{z} | \bar{z} \subseteq \bar{x}_1 + \ldots + \bar{x}_m\},$$
$$g(\bar{x}_1,\ldots,\bar{x}_n) = \overline{x_1 x_2 \ldots x_n}.$$

It can be verified obviously that $(\bar{R},f,g)$ is a Krasner $(m,n)-$ary hyperring.

Example 1.4. Let $(H,\leq,+)$ be a totally ordered group and $x_1^m \in H$. Set $k \in \{i | x_i = max\{x_1,\ldots,x_n\}\}$ and $c = card\{i | x_i = max\{x_1,\ldots,x_n\}\}$. Now, let

$$f(x_1,\ldots,x_m) = \begin{cases} \{t \in H | t \leq x_k\}, & \text{if } c > 1 \\ x_k, & \text{if } c = 1. \end{cases}$$

It follows that (H,f) is a canonical $m-$ary hypergroup on H. If $(H,+,\cdot)$ is a ring, then it can be verified that $(H,f,\cdot)$ is a Krasner $(m,2)-$ary hyperring.

A commutative krasner $(m,n)-$ary hyperring (R,f,g), in which is valid: $g(a_1^n) = 0$ implies there exists $1 \leq j \leq n$ such that $a_j = 0$, is called an integral hyperdomain.

Let S be a non-empty subset of a Krasner $(m,n)-$hyperring (R,f,g). If (S,f,g) is a Krasner $(m,n)-$hyperring, then S is called a subhyperring of R.

Let I be a non-empty subset of R and $1 \leq i \leq n$. we call I an $i-$hyperideal of R if:

1. I is a subhypergroup of the canonical $m-$ary hypergroup (R,f), i.e., (I,f) is a canonical $m-$ary hypergroup.
2. For every $x_1^n \in R$, $g(x_1^{i-1}, I, x_{i+1}^n) \subseteq I$.

Also, if for every $1 \leq i \leq n$, I is an $i-$hyperideal, then I called a hyperideal of R. Every hyperideal of R is a subhyperring of R. A hyperideal of an integral hyperdomain R is called principal hyperideal of R, if it is generated by a single element, while R is called principal hyperideal domain, if it is an integral hyperdomain and every hyperideal of R is principal.

2. $(m,n)-$hypermodules

Definition 2.1. ([4]). Let M be a non-empty set. Then, $M = (M,h,k)$ is an $(m,n)-$*ary hypermodule* (or $(m,n)-$*hypermodules*) over an $(m,n)-$ary hyperring R, if (M,h) is an $m-$ary hypergroup and the map

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \longrightarrow \wp^*(M)$$

satisfies in the following conditions:

(1) $k(r_1^{n-1}, h(x_1^m)) = h(k(r_1^{n-1}, x_1), \ldots, k(r_1^{n-1}, x_m))$,

(2) $k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, x) = h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x), \ldots, k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x))$,

(3) $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, x) = k(r_1^{n-1}, k(r_m^{n+m-2}, x))$.

If k is a scalar $n-$ary hyperoperation, $S_1, \ldots, S_{n-1}$ subsets of R and $M_1 \subseteq M$, we set

$$k(S_1, \ldots, S_{n-1}, M_1) = \bigcup \{k(r_1, \ldots, r_{n-1}, x) \mid r_i \in S_i,\ i = 1, \ldots, n-1,\ x \in M_1\}.$$

An $(m,n)-$ary hypermodule M is an $R-$*hypermodule*, if $m = n = 2$.

Example 2.2. Let $M = \{0,1,2\}$ and $(R,f,g) = der_{(3,2)}(\mathbb{Z},+,\cdot)$ (see example 1.2). We define the commutative hyperoperations h and k as follows:

$$h(0,0,0) = h(0,0,2) = h(0,2,2) = h(2,2,2) = \{0,2\},$$

$$h(0,0,1) = h(0,2,1) = h(2,2,1) = 1,$$

$$h(0,1,1) = h(2,1,1) = \{0,2\},$$

$$h(1,1,1) = 1,$$

and $k : R \times M \to \wp^*(M)$,

$$k(r,x) = \begin{cases} \{0,2\} & \text{if } r \in 2\mathbb{Z} \text{ or } x \in \{0,2\} \\ 1 & \text{else} \end{cases}$$

Then (M,h,k) is a $(3,2)-$ary hypermodule over a $(3,2)-$ary hyperring (R,f,g).

Example 2.3. Let $(R,+,\cdot)$ be a hyperring and $(M,+)$ an $R-$hypermodule. If N is a subhypermodule of M then set:

$$
\begin{aligned}
h(x_1^m) &= \sum_{i=1}^{m} x_i + N, && \forall\ x_1^m \in M,\\
f(r_1^m) &= \sum_{i=1}^{m} r_i, && \forall\ r_1^m \in R,\\
g(x_1^n) &= \prod_{i=1}^{n} r_i, && \forall\ r_1^n \in R,\\
k(r_1^{n-1},x) &= (\sum_{i=1}^{n-1} r_i)\cdot x + N, && \forall\ r_1^{n-1} \in R,\ \forall x \in M.
\end{aligned}
$$

Then (M,h,k) is an $(m,n)-$ary hypermodule over $(m,n)-$ary hyperring (R,f,g).

A canonical $(m,n)-$ary hypermodule (namely canonical $(m,n)-$ary hypermodule) is an $(m,n)-$ary hypermodule with a canonical $m-$ary hypergroup (M,h) over a Krasner $(m,n)-$ary hyperring (R,f,g).

Example 2.4. Let R be a Krasner hyperring and $(M,+,\cdot)$ a canonical $R-$hypermodule. Then, R with $m-$ary hyperoperation $f(r_1^m)=\sum_{i=1}^{m} r_i$, and $n-$ary hyperoperation $g(r_1^n)=\prod_{i=1}^{n} r_i$, is a Krasner $(m,n)-$ary hyperring. Also, M with hyperoperation h with $h(x_1^m)=\sum_{i=1}^{m} x_i$, where $x_i \in M$, is a canonical $m-$ary hypergroup. Now, we define the scalar $n-$ary hyperoperation k with

$$k(r_1,\dots,r_{n-1},x) := (\prod_{1}^{n} r_i)\cdot x.$$

Then M is a canonical $(m,n)-$ary hypermodule over Krasner $(m,n)-$ary hyperring R.

Definition 2.5. Let (M,h,k) be an $(m,n)-$ary hypermodule over an $(m,n)-$ary hyperring R. A non-empty subset $N \subseteq M$ is called an $(m,n)-$*ary subhypermodule* of M, if (N,h,k) is an $(m,n)-$ary hypermodule over the $(m,n)-$ary hyperring R.

Let (M_1,h_1,k_1) and (M_2,h_2,k_2) be two $(m,n)-$ary hypermodules over an $(m,n)-$ary hyperring R. A *homomorphism* from M_1 to M_2 is a mapping $\phi : M_1 \longrightarrow M_2$ such that

(1) $\phi(h_1(a_1,\dots,a_m)) = h_2(\phi(a_1),\dots,\phi(a_m))$,

(2) $\phi(k_1(r_1,\dots,r_{n-1},a)) = k_2(r_1,\dots,r_{n-1},\phi(a))$.

Lemma 2.6. *Let (M_1,h_1,k_1) and (M_2,h_2,k_2) be two $(m,n)-$ary hypermodules over an $(m,n)-$ary hyperring R and $\phi : M_1 \longrightarrow M_2$ a homomorphism. Then*

(1) *If S is an $(m,n)-$ary subhypermodule of M_1 over an $(m,n)-$ary hyperring R, then $\phi(S)$ is an $(m,n)-$ary subhypermodule of M_2.*

(2) *If K is an $(m,n)-$ary subhypermodule of M_2 over an $(m,n)-$ary hyperring R, such that $\phi^{-1}(K) \neq \emptyset$, then $\phi^{-1}(K)$ is an $(m,n)-$ary subhypermodule of M_1.*

Proof. (1) We know $\phi(S)$ is an $m-$ary subhypergroup of M_2. Let $r_1, r_2, \ldots, r_{n-1} \in R$ and $y \in \phi(S)$, then there exists $x \in S$ such that $\phi(x) = y$. Hence $k(r_1, \ldots r_{n-1}, y) = k(r_1, \ldots, r_{n-1}, \phi(x)) = \phi(r_1, \ldots, r_{n-1}, x) \in \phi(S)$.

(2) The proof of this part is similar to (1). □

If X is an $(m,n)-$ary subhypermodule of a canonical $(m,n)-$ary hypermodule M, then $< X >$ is the $(m,n)-$ary subhypermodule generated by elements of X.

Definition 2.7. Let M_1^t be subhypermodules of a canonical $(m,n)-$ary hypermodule (M,h,k) and $t = l(m-1)+1$, then M is called a (internal) direct sum $M_1 \oplus \ldots \oplus M_t$ if satisfies the following axioms:

(1) $M = h_l(M_1, \ldots, M_t)$ and

(2) $M_i \cap h_l(M_1, \ldots, M_{i-1}, 0, M_{i+1}, \ldots M_t) = \{0\}$.

3. Free $(m,n)-$hypermodules

In the following in this section, an $(m,n)-$ary hypermodule is a canonical $(m,n)-$ary hypermodule over Krasner $(m,n)-$ hyperring.

Let (M,h,k) be an $(m,n)-$ary hypermodule over the $(m,n)-$ary hyperring R. Then

Definition 3.1. A linear combination of family $A = \{x_i : i \in I\}$ of elements of M is a sum of the form $h(k(r_{11}^{1(n-1)}, x_1), \ldots, k(r_{l1}^{l(n-1)}, x_l), \overset{(m-l)}{0})$ with $l \leq m$ and if $l > m, l = t(m-1)+1$, a linear combination of A is the form of

$$\underbrace{h(h(\ldots,h(h}_{t}(k(r_{11}^{1(n-1)}, x_1), \ldots, k(r_{m1}^{m(n-1)}, x_m)), k(r_{(m+1)1}^{(m+1)(n-1)}, x_{m+1}), \ldots,$$
$$k(r_{(2m-1)1}^{(2m-1)(n-1)}, x_{2m-1})), \ldots), k(r_{((l-1)(m-1)+2)1}^{((l-1)(m-1)+2)(n-1)}, x_{(l-1)(m-1)+2}), \ldots,$$
$$k(r_{(l(m-1)+1)1}^{(l(m-1)+1)(n-1)}, x_{l(m-1)+1}))),$$

where $r_{ij} \in R$ and set $\{r_{ij} : r_{ij} \neq 0\}$ is finite.

A linear combination of family $\{x_i : i \in I\}$ of elements of M is a sum of the form $h(k(r_{11}^{1(n-1)}, x_1), \ldots, k(r_{l1}^{l(n-1}, x_l))$. $\{x_i : i \in I\}$ is linear dependent if there exists a linear combination $h(k(r_{11}^{1(n-1)}, x_1), \ldots, k(r_{l1}^{l(n-1)}, x_l))$ containing 0, without begin all the r_{ij} equal to 0. Otherwise $\{x_i : i \in I\}$ is called linear independent.

Definition 3.2. A subset X of M generates M if every element of M belongs to a linear combination of elements from X.

Definition 3.3. Let (N,h,k) be an $(m,n)-$ary hypermodule over the $(m,n)-$ary hyperring R and let X a subset of N. Then X generates N freely if

(i) X generates N;

(ii) for every function ψ of X into an $(m,n)-$ary hypermodule M there exists a homomorphism $\varphi : N \to M$ such that $\varphi(x) = \{\psi(x)\}$ for every $x \in X$.

We call an $(m,n)-$ary hypermodule N is free, if there exists a subset X of N which generates N freely. Any set which freely generates N is called a free basis of N.

Let us construct now a free $(m,n)-$ary hypermodule over an $(m,n)-$ary hyperring (R,f,g). For this purpose we consider a non void set Ω and the set R^{Ω} of the functions with domain Ω and rang R. Next we choose from R^{Ω} all these functions which vanish almost every where. Denote this set by $E(\Omega)$. Then $E(\Omega)$ becomes a canonical $m-$ary hypergroup if we introduce in it an $m-$ary hyperoperation h defined as follows:

$$h(l_1^m) = \{d \in E(\Omega) : (\forall x \in \Omega), d(x) \in f(l_1(x), \ldots, l_m(x))\}.$$

We define a map

$$k' : \underbrace{\wp^*(R) \times \ldots \times \wp^*(R)}_{n-1} \times E(\Omega) \to \wp^*(E(\Omega))$$

as follow:

$$k'(A_1^n, l) = \{s \in E(\Omega) : (\forall x \in \Omega), s(x) \in \bigcup_{a_1 \in A_1, \ldots, a_{n-1} \in A_{n-1}} \{g(a_1^{n-1}, l(x))\}\}.$$

Also, we introduce an $n-$ary scalar hyperoperation k from $\underbrace{R \times \ldots \times R}_{n-1} \times E(\Omega)$ to $E(\Omega)$, with $k(a_1^{n-1}, l) = k'(\{a_1\}, \ldots, \{a_{n-1}\}, l)$. So, if $A \subseteq R$, then $k(A_1^{n-1}, l) = k'(A_1^{n-1}, l)$.

Theorem 3.4. *$E(\Omega)$ endowed with the $m-$ary hyperoperation h and with the scalar $n-$ary hyperoperation k become an $(m,n)-$ary hypermodule over the $(m,n)-$ary hyperring (R,f,g).*

Proof. The neutral element of h is the zero function, since f is well-defined and associative and commutative so h is well-defined and associative and commutative. Suppose now that $l \in h(l_1^m)$, then $l(x) \in f(l_1(x), \ldots, l_m(x))$ for every $x \in \Omega$, thus

$$l_i(x) \in f(-l_{i-1}(x), \ldots, -l_1(x), l(x), -l_m(x), \ldots, -l_{i+1}(x))$$

for every $x \in \Omega$, so $l_i \in h(-l_{i-1},\dots,-l_1,l,-l_m,\dots,-l_{i+1})$. Next for every $r_i, s_i \in R$ and $l, l_i \in E(\Omega)$ we have:

$$\begin{aligned} k(r_1^{n-1}, h(l_1^m)) &= k(r_1^{n-1}, \{d \in E(\Omega) : (\forall x \in \Omega), d(x) \in f(l_1(x),\dots,l_m(x))\}) \\ &= \{k(r_1^{n-1}, d) \in E(\Omega) : (\forall x \in \Omega), d(x) \in f(l_1(x),\dots,l_m(x))\} \\ &= h(k(r_1^{n-1}, l_1),\dots,k(r_1^{n-1}, l_m)). \end{aligned}$$

In addition,

$$\begin{aligned} &k(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, l) = k'(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, l) \\ &= \{s \in E(\Omega) : (\forall x \in \Omega), s(x) \in \bigcup_{a_j \in r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, 1 \le j \le n-1} \{g(a_1^{n-1}, l(x))\}\} \\ &= \{s \in E(\Omega) : (\forall x \in \Omega), s(x) \in g(r_1^{i-1}, f(s_1^m), r_{i+1}^{n-1}, l(x))\} \\ &= \{s \in E(\Omega) : (\forall x \in \Omega), s(x) \in f(g(r_1^{i-1}, s_1, r_{i+1}^{n-1}, l(x)),\dots,g(r_1^{i-1}, s_m, r_{i+1}^{n-1}, l(x)))\} \\ &= h(k(r_1^{i-1}, s_1, r_{i+1}^{n-1}, l),\dots,k(r_1^{i-1}, s_m, r_{i+1}^{n-1}, l)). \end{aligned}$$

In the similar way, we can prove that $k(r_1^{i-1}, g(r_i^{i+n-1}), r_{i+m}^{n+m-2}, l) = k(r_1^{n-1}, k(r_m^{n+m-2}, l))$ and therefore (M,h,k) is an (m,n)−ary hypermodule. □

Theorem 3.5. *There exists a subset B of $E(\Omega)$ linear independent, which generates $E(\Omega)$ and such as cardB=cardΩ.*

Proof. For every $a \in \Omega$, consider the function

$$f_a(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } x \neq a. \end{cases}$$

The set B of the functions f_a generates $E(\Omega)$. Also B is linear independent since if $0 \in h(k(r_{11}^{1(n-1)}, f_{a_1}),\dots,k(r_{l1}^{l(n-1)}, f_{a_l}))$, Then, for $1 \le i \le l$, we have

$$0 \in h(k(r_{11}^{1(n-1)}, f_{a_1}(a_i)),\dots,k(r_{l1}^{l(n-1)}, f_{a_l}(a_i))) = k(r_{i1}^{i(n-1)}, 1)$$

and also for every $1 \le i \le l$, there exists $1 \le j \le n-1$ such that $r_{ij} = 0$. Obviously card B=card Ω. □

Definition 3.6. An element x of an (m,n)−ary hypermodule M is called torsion free if $k(r_1^{n-1}, x) = 0$ implies there exists $1 \le j \le n-1$ such that $r_j = 0$. M is called torsion free if all its elements are torsion free.

In the following, in this paper, let $-k(r_1,\dots,r_{n-1},x) = k(-r_1,\dots,-r_{n-1},x)$, for every $r_1^{n-1} \in R$ and $x \in M$.

Theorem 3.7. *Let N be an $(m,n)-$ary hypermodule over the unitary $(m,n)-$ary hyperring (R,f,g) and let $B=\{e_1,\dots,e_m\}$ be a finite subset of N. Then the following are equivalent:*

(1) *B is a free basis of N;*

(2) *B is linear independent and generates N;*

(3) *every e_i is torsion free and $N=<e_1>\oplus\dots\oplus<e_m>$.*

Proof. (1) ⇒ (2). Consider the $(m,n)-$ary hypermodule $E(B)$. Then, because of the definition, for the function

$$\psi: B\to E(B),\qquad \psi(e_i)=f_{e_i}\ \ (i=1,\dots,m)$$

there exists a homomorphism

$$\varphi: N\to E(B),\qquad \varphi(e_i)=\{f_{e_i}\}\ \ (i=1,\dots,m).$$

Now, if $0\in h(k(r_{11}^{1(n-1)},e_1),\dots,k(r_{m1}^{m(n-1)},e_m))$, then

$$0=\varphi(0)\in\varphi(h(k(r_{11}^{1(n-1)},e_1),\dots,k(r_{m1}^{m(n-1)},e_m)))=h(k(r_{11}^{1(n-1)},\varphi(e_1)),\dots,k(r_{m1}^{m(n-1)},$$

$\varphi(e_m)))=h(k(r_{11}^{1(n-1)},f_{e_1}),\dots,k(r_{m1}^{m(n-1)},f_{e_m}))$, and since the $f_{e_i}(i=1,\dots,m)$ are linear independent for every $1\le i\le m$, there exists $1\le j\le n-1$ such that $r_{ij}=0$.

(2) ⇒ (3). If $k(r_1^{n-1},e_i)=0$, then there exists $1\le j\le n-1$ such that $r_j=0$, since B is linear independent. It is obvious $x\in h(<e_1>,\dots,<e_m>)$. Suppose that $x\in<e_i>\cap h(<e_1>,\dots<e_{i-1}>,0,<e_{i+1}>,\dots,<e_m>)$. Then $x=k(s_{i1}^{i(n-1)},e_i)$ and $x\in h(k(s_{11}^{1(n-1)},e_1),\dots,k(s_{(i-1)1}^{(i-1)(n-1)},e_{i-1}),0,k(s_{(i+1)1}^{(i+1)(n-1)},e_{i+1}),\dots,k(s_{m1}^{1(n-1)},e_m))$. Thus $0\in h(x,-x,\overset{(m-2)}{0})\subseteq h(k(s_{11}^{1(n-1)},e_1),\dots,-k(s_{i1}^{i(n-1)},e_i),\dots,k(s_{m1}^{1(n-1)},e_m))$, so for every $1\le i\le m$, there exists $1\le j\le n-1$ such that $s_{ij}=0$ and therefore the conclusion.

(3) ⇒ (2). It is a clear that B generates N. Now, let B is linear dependent. Thus in the relation $0\in h(k(r_{11}^{1(n-1)},e_1),\dots,k(r_{m1}^{m(n-1)},e_m))$, not all the coefficients are zero, and let $k(r_{i1}^{i(n-1)},e_i)\in-(h(k(r_{11}^{1(n-1)},e_1),\dots,k(r_{(i-1)1}^{(i-1)(n-1)},e_{i-1}),0,k(r_{(i+1)1}^{(i+1)(n-1)},e_{i+1}),\dots,k(r_{m1}^{1(n-1)},e_m)))$. Then

$$k(r_{i1}^{i(n-1)},e_i)\in<e_i>\cap h(<e_1>,\dots<e_{i-1}>,0,<e_{i+1}>,\dots,<e_m>)=\{0\}.$$

and therefore there exists $1\le j\le n-1$ such that $r_j=0$. Since the e_i are torsion free, which contradicts again linear independent B.

(2) ⇒ (1). For every $x\in N$ there are $s_{ij}\in R$ such as

$$x\in h(k(s_{11}^{1(n-1)},e_1),\dots,k(s_{m1}^{m(n-1)},e_m)).$$

We define functions $f_1,\dots,f_m$ from N to R as follows:

$$f_i(x)=s_{ii},\qquad (i=1,\dots,m)$$

$$f_i(e_j)=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

and $f_i(h(x_1^m))=f(f_i(x_1),\ldots,f_i(x_m)), f_i(k(r_1^{n-1},x))=g(r_1^{n-1},f_i(x)), x,x_1^m\in N, r_1^m\in R$. If M is an $(m,n)-$ary hypermodule and ψ a function from B to M, we define a homomorphism φ from N to M as follows:

$\varphi(x)=h(k(\overset{(n-2)}{1_R},f_1(x),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f_m(x),\psi(e_m)))$ for every $x\in N$. φ satisfies the condition $\varphi(e_i)=\{\psi(e_i)\}$ and it is indeed a homomorphism since:

$$\begin{aligned}\varphi(h(x_1^m)) &= h(k(\overset{(n-2)}{1_R},f_1(h(x_1^m)),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f_m(h(x_1^m)),\psi(e_m)))\\
&= h(k(\overset{(n-2)}{1_R},f(f_1(x_1),\ldots,f_1(x_m)),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f(f_m(x_1),\ldots,f_m(x_m)),\psi(e_m)))\\
&= h(h(k(\overset{(n-2)}{1_R},f_1(x_1),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f_1(x_m),\psi(e_1))),\ldots,h(k(\overset{(n-2)}{1_R},f_m(x_1),\psi(e_m)),\\
&\quad \ldots,k(\overset{(n-2)}{1_R},f_m(x_m),\psi(e_m))))\\
&= h(h(k(\overset{(n-2)}{1_R},f_1(x_1),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f_m(x_1),\psi(e_m))),\ldots,h(k(\overset{(n-2)}{1_R},f_1(x_m),\psi(e_m)),\\
&\quad \ldots,k(\overset{(n-2)}{1_R},f_m(x_m),\psi(e_m))))\\
&= h(\varphi(x_1),\ldots,\varphi(x_m)).\end{aligned}$$

In the similar way, we can prove that, $\varphi(k(r_1^{n-1},x_1))=k(r_1^{n-1},\varphi(x_1))$ and therefore φ is homomorphism. Now let φ' be another homomorphism from N to M such as $\varphi'(e_i)=\psi(e_i)(i=1,\ldots,m)$. Then, for every $x\in N$

$$\begin{aligned}& x\in h(k(\overset{(n-2)}{1_R},f_1(x),e_1))\ldots,k(\overset{(n-2)}{1_R},f_m(x),e_m))\\
&\Rightarrow \varphi'(x)\subseteq\varphi'(h(k(\overset{(n-2)}{1_R},f_1(x),e_1),\ldots,k(\overset{(n-2)}{1_R},f_m(x),e_m)))\\
&\Rightarrow \varphi'(x)\subseteq h(k(\overset{(n-2)}{1_R},f_1(x),\psi(e_1)),\ldots,k(\overset{(n-2)}{1_R},f_m(x),\psi(e_m)))\Rightarrow\varphi'(x)\subseteq\varphi(x).\end{aligned}$$

Therefore φ is the maximum homomorphism having the property which the definition of the free $(m,n)-$ary hypermodule demands. □

Example 3.8. Let $(H,.)$ be a commutative almost group (i.e. a semigroup $H=H^*\cup\{0\}$, where $(H^*,.)$ is a group and 0 a two side absorbing element). Now let $g(x_1^n)=\prod_{i=1}^{n}x_i$, then (H,g) is an $n-$ary group. For every $x_1^k\in H^*$, we define an $m-$ary hyperoperation f on H as follows:

$$f(x_1^k, \overset{(m-k)}{0}) = \begin{cases} 0 & k=0 \\ \bigcup_{i=1}^{k}\{x_i\} & |\bigcup_{i=1}^{k} x_i| = k \\ H-\{x_1\} & k=2, |\bigcup_{i=1}^{k} x_i| = 1 \\ H & k \geq 3, |\bigcup_{i=1}^{k} x_i| < k, \end{cases}$$

f is a commutative hyperoperation and 0 is a scaler identity and $f(\overset{(m)}{0}) = 0$. Then the hyperstructure (H, f, g) is an $(m,n)-$ary hyperfield and therefore (H, f, g) is a free $(m,n)-$ary hypermodule over an $(m,n)-$ary hyperfield (H, f, g).

Lemma 3.9. *Let M be an $(m,n)-$ary hypermodule, N a finitely generated free $(m,n)-$ary hypermodule and $\theta : M \to N$ an epimorphism. Then M has a subhypermodule F isomorphic to N such as $M = F \oplus ker\theta$.*

Proof. Let $\{e_1, \ldots, e_m\}$ be a basis of N. Since θ is an epimorphism, there are $n_i \in M$ such as $\theta(n_i) = e_i$, $(i = 1, \ldots, m)$. Consider a function $\psi : B \to M$ with $\psi(e_i) = n_i$. Since N is free, there is a homomorphism $\varphi : N \to M$ such as $\varphi(e_i) = \{\psi(e_i)\} = \{n_i\}, (i = 1, \ldots, m)$. Then $\theta(\varphi(e_i)) = \theta(n_i) = e_i$. So if $x \in h(k(x_{11}^{1(n-1)}, e_1), \ldots, k(x_{m1}^{m(n-1)}$
$, e_m))$, then $\theta(\varphi(x)) = \theta(h(k(x_{11}^{1(n-1)}, n_1), \ldots, k(x_{m1}^{m(n-1)}, n_m))) = h(k(x_{11}^{1(n-1)}, e_1),$
$\ldots, k(x_{m1}^{m(n-1)}, e_m)) \ni x$. Thus $x \in \theta(\varphi(x))$ for every $x \in N$. By Theorem 2.6, $F = \varphi(N)$ is a subhypermodule of M. Then $\theta(F) = \theta(\varphi(N)) \supseteq N$, but $\theta(F) \subseteq N$, thus $\theta(F) = N$. If $y \in M$, then there is an element $y' \in F$ such as $\theta(y) = \theta(y')$, from where, $0 \in h(\theta(y), -\theta(y'), \overset{(m-2)}{0})$, or $0 \in \theta(h(y, -y', \overset{(m-2)}{0}))$, or $h(y, -y', \overset{(m-2)}{0}) \cap ker\theta \neq \emptyset$, or $y \in h(F, ker\theta, \overset{(m-2)}{0})$. Thus $M = h(F, ker\theta, \overset{(m-2)}{0})$. Now, let $t \in F \cap ker\theta$. Since $t \in F$, there is an element $x \in N$ such as $t \in \varphi(x)$ and since $t \in ker\theta$, $\theta(t) = 0$. Let $x \in h(k(x_{11}^{1(n-1)}, e_1), \ldots, k(x_{m1}^{m(n-1)}, e_m))$, then

$$\begin{aligned} 0 = \theta(t) \in \theta(\varphi(x)) &= f(h(k(x_{11}^{1(n-1)}, n_1), \ldots, k(x_{m1}^{m(n-1)}, n_m)) \\ &= h(k(x_{11}^{1(n-1)}, e_1), \ldots, k(x_{m1}^{m(n-1)}, e_m)), \end{aligned}$$

which means that for every $1 \leq i \leq m$, there exists $1 \leq j \leq n-1$ such that $x_{ij} = 0$. So $x = 0$, also $t = 0$ and therefore $\ker\theta \cap F = \{0\}$. □

Theorem 3.10. *Let M be a finitely generated free $(m,n)-$ary hypermodule over a unitary principal hyperideal domain R and let N be a subhypermodule of M. Then N is also free and dim $N \leq$ dim M.*

Proof. Let $B = \{x_1,\ldots,x_m\}$ be a basis of M and let $N_j = N\cap < x_1,\ldots,x_j >$, $j = 0,1,\ldots,m$. $N_0 = \{0\}$ and $N_m = N$. $N_1 = N\cap < x_1 >$ is a subhypermodule of $< x_1 >$. The set $A = \{a : a \in R, k(\overset{(n-2)}{1_R}, a, x_1) \in N_1\}$ is a hyperideal of R and let a_1 be its generator. Thus $N_1 =< k(\overset{(n-2)}{1_R}, a, x_1) >$ and therefore N_1 is $\{0\}$ or it has a basis with only one element. Suppose now that N_t has a free basis and that $dim N_t \le t$. Consider the set $A = \{a : a \in R$ and there exists $x \in N, b_{11}^{1(n-1)},\ldots,b_{t1}^{t(n-1)} \in R$ such that $x \in h(k(b_{11}^{1(n-1)},x_1),\ldots,k(b_{t1}^{t(n-1)},x_t),k(\overset{(n-2)}{1_R},a,x_{t+1}),\overset{(m-t-1)}{0})\}$. A is a hyperideal, because if $a_1^m \in A$, then there are $y_1^m \in N$ such as $y_i \in h(k(b_{i11}^{i1(n-1)},x_1),\ldots,k(b_{it1}^{it(n-1)},x_t),k(\overset{(n-2)}{1_R},a_i,x_{t+1}),\overset{(m-t-1)}{0})$, from where we get

$$\begin{aligned}
h(y_1^m) \in h(&h(k(b_{111}^{11(n-1)},x_1),\ldots,k(b_{1t1}^{1t(n-1)},x_t),k(\overset{(n-2)}{1_R},a_1,x_{t+1}),\overset{(m-t-1)}{0}),\ldots,\\
&h(k(b_{m11}^{m1(n-1)},x_1),\ldots,k(b_{mt1}^{mt(n-1)},x_t),k(\overset{(n-2)}{1_R},a_m,x_{t+1}),\overset{(m-t-1)}{0}))\\
= h(&k(s_{11}^{1(n-1)},x_1),k(s_{t1}^{t(n-1)},x_t),k(\underbrace{f(\overset{(m)}{1_R}),\ldots,f(\overset{(m)}{1_R})}_{n-2},f(a_1^m),x_{t+1})),
\end{aligned}$$

where $s_{ij} = f(b_{1ij},\ldots,b_{mij})$. Thus $f(a_1^m) \in A$. The remaining condition can be easily proved, so A is indeed a hyperideal. Let a_{t+1} be a the generator of A. If $a_{t+1} = 0$ then $N_{t+1} = N_t$ and induction is complete. If $a_{t+1} \neq 0$, consider $y \in N_{t+1}$ such as $y \in h(k(a_{11}^{1(n-1)},x_1), k(a_{t1}^{(t(n-1)},x_t), k(\overset{(n-2)}{1_R},a_{t+1},x_{t+1}),\overset{(m-t-1)}{0})$. For every $x \in N_{t+1}$, there is $c_1^{n-1} \in R$ such as $h(x,-k(c_1^{n-1},y),\overset{(m-2)}{0}) \cap N_t \neq \emptyset$. Thus $N = h(N_t,< y >,\overset{(m-2)}{0})$. But $N_t\cap < y >= \{0\}$, so $N = N_t \oplus < y >$ and the induction is complete. □

Theorem 3.11. *Every finitely generated torsion free (m,n)−ary hypermodule over a unitary principal hyperideal domain R is free with a finite basis.*

Proof. Let M be a finitely generated torsion free (m,n)−ary hypermodule and let $C = \{s_1,\ldots, s_m\}$ be a set which generates M. Consider the maximum subset $B = \{s_1,\ldots,s_l\}$ of linear independent elements of C and let N be the subhypermodule of M generated by B. Then N, because of Lemma 3.7, is free. According to our hypothesis every set of the form $\{s_1,\ldots,s_l,s_i\}$, $l < i \le m$ is linear dependent. Thus in the relation $0 \in h(k(r_{11}^{1(n-1)},s_1),\ldots,k(r_{l1}^{l(n-1)},s_l), k(t_{i1}^{i(n-1)},s_i),\overset{(m-(l+1))}{0})$, not all the coefficients are zero, and more precisely the $t_{i1}^{i(n-1)} \neq 0$, since $s_1,\ldots,s_l$ are linear independent. Now $k(t_{i1}^{i(n-1)},s_i) \in -(h(k(r_{11}^{1(n-1)},s_1),\ldots,k(r_{l1}^{l(n-1)},s_l),\overset{(m-l)}{0}))$ so $k(t_{i1}^{i(n-1)}, s_i) \in N$ for every $i > l$. Let $t = t_{(l+1)1}^{(l+1)(n-1)},\ldots,t_{m1}^{m(n-1)}$, then $k(t,s_i) \in N$ for every $i \le m$, so $k(t,x) \in N$, for every $x \in M$ and therefore the map $\theta : x \to k(t,x)$ is a homomorphism from M to a subhypermodule of N. But, since M is torsion free the kernel of f is the $\{0\}$, thus M is isomorphic to a subhypermodule T of N and by Theorem 3.10, T is free (m,n)−ary hypermodule and therefore M is free (m,n)−ary hypermodule. □

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