TAMKANG JOURNAL OF MATHEMATICS
Volume 45, Number 3, 243-250, September 2014
doi:10.5556/j.tkjm.45.2014.876
Available online at http://journals.math.tku.edu.tw/

APPROXIMATION OF FUNCTIONS OF CLASS $Lip(\alpha,r)$, $(r\geq 1)$, BY $(N,p_n)(E,1)$ SUMMABILITY MEANS OF FOURIER SERIES

SHYAM LAL AND ABHISHEK MISHRA

Abstract. In this paper, two new theorems on degree of approximation of a function $f\in Lip(\alpha,r)$, $(r\geq 1)$, have been established. A new technique is applied to find the estimate.

1. Introduction

Chandra [1], Qureshi [6, 7], Khan [3] Mohapatra and Russell [5], Sahney and Rao [8] have determined the degree pf approximation of functions of the class $Lip\alpha$ and $Lip(\alpha,r)$, $0<\alpha\leq 1$, $r\geq 1$. Most of these results are not satisfactory for $\alpha=1$ in the sense that the estimates for $r>1$ and $\alpha=1$ are not of $O(n^{-1})$. Therefore this deficiency for determine the degree of approximation has motivated to investigate the degree of approximation using generalized Minkowski inequality considering cases $0<\alpha<1$ and $\alpha=1$ separately. It is important to note that till now no work seems to have been done to obtain the degree of approximation of a function $f\in Lip(\alpha,r)$ class by product summabiliy means of the form $(N,p_n)(E,1)$. In an attempt to make an advance study in this direction, in this paper, the estimates of degree of approximation of the function $f\in Lip(\alpha,r)$ class by $(N,p_n)(E,1)$ means of Fourier series have been determined. This estimate is new, better and sharper than all previously known estimates.

2. Definitions and Notations

Let $f(x)$ be a 2π-periodic function, Lebesgue integrable on $[0,2\pi]$ and belonging to $Lip(\alpha,r)$, $(r\geq 1)$ class. The Fourier series of $f(x)$ is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(x) \tag{2.1}$$

Received March 25, 2011, accepted December 24, 2013.
2010 *Mathematics Subject Classification*. 42B05, 42B08.
Key words and phrases. Trigonometric approximation, Fourier series, regular (N,p_n) means, $(E,1)$ means and $(N,p_n)(E,1)$ summability means and generalized Minkowski inequality.
Corresponding author: Shyam Lal.

with partial sums $s_n(f;x)$.

A function $f(x)$ is said to belong to the class $Lip\alpha$ if

$$\left|f(x+t)-f(x)\right| = O(|t|^{\alpha})\ for\ 0<\alpha\leq 1.$$

$f(x)$ is said to belong to the class $Lip(\alpha,r)$ for $0<\alpha\leq 1,\ r\geq 1$ if

$$\left\{\int_o^{2\pi}\left|f(x+t)-f(x)\right|^r dx\right\}^{1/r} = O(|t|^{\alpha}) \quad \text{(def 5.38 of McFadden [4])}.$$

We define the norm $\|\ \|_r$ by $\|f\|_r = \frac{1}{2\pi}\left\{\int_0^{2\pi}\left|f(x)\right|^r dr\right\}^{1/r}, r\geq 1,$
and the degree of approximation $E_n(f)$ be given by

$$E_n(f) = min\|f-t_n\|_r$$

where $t_n(x) = \dfrac{1}{2}a_0 + \sum_{\nu=1}^{n}(a_\nu\cos\nu x + b_\nu\sin\nu x)$ is n^{th} degree trigonometric polynomial (Zygmund [10], p.114) .

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having n^{th} partial sum $s_n = \sum_{\nu=0}^{n} u_\nu$. Let $\{p_n\}$ be a sequence of constants, real of complex valued and let

$$P_n = p_0+p_1+p_2+\cdots+p_n,\ p_0>0,\ P_n\neq 0,$$

and define t_n^N by $t_n^N = \dfrac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \dfrac{1}{P_n}\sum_{k=0}^{n} p_k s_{n-k}$. The $\{t_n^N\}$ is defined as the sequence of Nörlund means of sequence $\{s_n\}$ generated by the sequence of constants $\{p_n\}$. If $t_n^N\to s$ when $n\to\infty$, we say that $\sum_{n=0}^{\infty} u_n$ is summable by Nörlund means or summable (N,p_n) to the sum s.

Let $E_n^{(1)} = \dfrac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k$. If $E_n^{(1)}\to s$ as $n\to\infty$, then $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by the Euler Method (E,1). (Hardy [2]).

The (N,p_n) transform of $(E,1)$ transform defines the $(N,p_n)(E,1)$ transform t_n^{NE} of the partial sum s_n of the series $\sum_{n=0}^{\infty} u_n$ by

$$\begin{aligned} t_n^{NE} &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}E_k^{(1)} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\frac{1}{2^k}\sum_{k=0}^{\nu}\binom{k}{\nu}s_\nu \\ &= \frac{1}{P_n}\sum_{k=0}^{n} p_k\frac{1}{2^{n-k}}\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}s_\nu. \end{aligned}$$

If $t_n^{NE}\to s$ as $n\to\infty$, $\sum_{n=0}^{\infty} u_n$ is said to be summable $(N,p_n)(E,1)$ to s. If the method of summability (N,p_n) is superimposed on $(E,1)$ method of summabilty, $(N,p_n)(E,1)$ summability is obtained.

We write, $\phi(x,t)=f(x+t)+f(x-t)-2f(x)$,

$$(NE)_n(t)=\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

3. Theorems

We prove the following theorems:

Theorem 1. *Let (N,p_n) be a regular Nörlund method defined by a positive sequence $\{p_n\}$ such that*

$$\sum_{k=0}^{n}\left|\Delta p_k\right|=O\left(\frac{P_n}{(n+1)}\right). \tag{3.1}$$

Let $f:\mathbb{R}\to\mathbb{R}$ is 2π-periodic, Lebesgue integrable on $[0,2\pi]$ and belonging to the class $Lip(\alpha,r)$, $(r\geq 1)$, then its degree of approximation by $(N,p_n)(E,1)$ means

$$t_n^{NE}=\frac{1}{P_n}\sum_{k=0}^{n}p_k\frac{1}{2^{n-k}}\sum_{\nu=0}^{n-k}\binom{n-k}{\nu}s_\nu$$

of the Fourier series (2.1) satisfies for $n=0,1,2,3,\ldots$

$$\|t_n^{NE}-f\|_r=\begin{cases}O\left((n+1)^{-\alpha}\right) & ,0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right) & ,\ \alpha=1.\end{cases}$$

Theorem 2. *Let (N,p_n) be a regular Nörlund method generated by a positive monotonic sequence $\{p_n\}$ satisfying*

$$(n+1)p_n=O(P_n), \tag{3.2}$$

Then the degree of approximation of $f\in Lip(\alpha,r)$, $r\geq 1$ by $(N,p_n)(E,1)$ means t_n^{NE} of its Fourier series (2.1) is given by

$$\|t_n^{NE}-f\|_r=\begin{cases}O\left((n+1)^{-\alpha}\right) & ,0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right) & ,\ \alpha=1.\end{cases}$$

4. Lemmas

We need following lemmas for the proof of our theorem.

Lemma 4.1. $(NE)_n(t)=O(n+1)$, $for\ 0<t\leq\frac{\pi}{(n+1)}$.

Proof. For $0<t\leq\frac{\pi}{(n+1)}$, $\sin nt\leq nt$, and $\sin(t/2)\geq(t/\pi)$, we have

$$|(NE)_n(t)|=\left|\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right|$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \frac{(n-k+1)}{t/\pi} \\
&= \frac{(n+1)}{4P_n}\sum_{k=0}^{n} p_k \\
&= O(n+1).
\end{aligned}
$$

Lemma 4.2. $(NE)_n(t) = O\left(\frac{1}{(n+1)t^2}\right)$, $for\ \frac{\pi}{(n+1)} < t \leq \pi.$

Proof. For $\frac{\pi}{(n+1)} < t \leq \pi$ using $\sin(t/2) \geq (t/\pi)$, $|\sin nt| \leq 1$ and Abel's lemma, we have

$$
\begin{aligned}
|(NE)_n(t)| &= \left| \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right| \\
&\leq \frac{1}{2tP_n}\sum_{k=0}^{n}\left| p_k \cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)\right| \\
&\leq \frac{1}{2tP_n}\left[\sum_{k=0}^{n-1}\left|(p_k-p_{k+1})\right|\sum_{r=0}^{k}\cos^{k-r}\left(\frac{t}{2}\right)\sin(k-r+1)\left(\frac{t}{2}\right)\right. \\
&\qquad \left. +p_n\sum_{k=0}^{n}\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)\right] \\
&\leq \frac{1}{2t^2P_n}\left[\sum_{k=0}^{n-1}\left|\Delta p_k\right| + \left|p_n\right|\right] \\
&\leq \frac{1}{2t^2P_n}\sum_{k=0}^{n}\left|\Delta p_k\right| = O\left(\frac{1}{(n+1)t^2}\right).
\end{aligned}
$$

Lemma 4.3. *If $f \in Lip(\alpha, r)$, $0 < \alpha \leq 1$, $r \geq 1$, then*

$$
\left[\int_0^{2\pi}\left|\phi(x,t)\right|^r dx\right]^{1/r} = O\left(|t|^{\alpha}\right)
$$

Proof. Clearly,

$$
\begin{aligned}
\left|\phi(x,t)\right| &= \left|f(x+t)+f(x-t)-2f(x)\right| \\
&\leq \left|f(x+t)-f(x)\right| + \left|f(x-t)-f(x)\right|.
\end{aligned}
$$

Then, using Minkowski's inequality, we have

$$
\begin{aligned}
\left[\int_0^{2\pi}\left|\phi(x,t)\right|^r dx\right]^{\frac{1}{r}} &\leq \left[\int_0^{2\pi}\left\{\left|f(x+t)-f(x)\right| + \left|f(x-t)-f(x)\right|\right\}^r dx\right]^{\frac{1}{r}} \\
&\leq \left[\int_0^{2\pi}\left|f(x+t)-f(x)\right|^r dx\right]^{\frac{1}{r}} + \left[\int_0^{2\pi}\left|f(x-t)-f(x)\right|^r dx\right]^{\frac{1}{r}} \\
&= O(|t|^{\alpha}) + O(|t|^{\alpha}) = O(|t|^{\alpha}).
\end{aligned}
$$

5. Proof of Theorem 1

Follwing Titchmarsh [9], $s_n(f;x)$ of Fourier series (2.1) is given by

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(x,t)\frac{\sin(n+1/2)\,t}{\sin(t/2)}dt$$

Denoting $(E,1)$ means of $s_n(f;x)$ by $E_n^{(1)}(f;x)$, we have

$$\frac{1}{2^n}\sum_{k=o}^{n}\binom{n}{k}\{s_n(f;x)-f(x)\} = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi(x,t)}{\sin(t/2)}\sum_{k=o}^{n}\binom{n}{k}\sin\left(k+\frac{1}{2}\right)tdt$$

$$\frac{1}{2^n}\sum_{k=o}^{n}\binom{n}{k}s_n(f;x)-f(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi(x,t)}{\sin(t/2)}\, Im\left\{\sum_{k=o}^{n}\binom{n}{k}e^{i(k+\frac{1}{2})t}\right\}dt$$

$$or,\ E_n^{(1)}(f;x)-f(x) = \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi(x,t)}{\sin(t/2)}\, Im\left\{e^{it/2}\sum_{k=o}^{n}\binom{n}{k}e^{ikt}\right\}dt$$

$$= \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi(x,t)}{\sin(t/2)}\, Im\left\{e^{it/2}(1+e^{it})^n\right\}dt$$

$$= \frac{1}{2^{n+1}\pi}\int_0^{\pi}\frac{\phi(x,t)}{\sin(t/2)}\, Im\left\{2^n\cos^n\left(\frac{t}{2}\right)e^{i(n+1)t/2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{\cos^n\left(\frac{t}{2}\right)\sin(n+1)\left(\frac{t}{2}\right)}{\sin(t/2)}dt.$$

(N,p_n) means of $E_n^{(1)}(f;x)$ i.e. $t_n^{NE}(f;x)$ is given by

$$\frac{1}{P_n}\sum_{k=o}^{n}p_k\{E_n^{(1)}(f;x)-f(x)\} = \frac{1}{P_n}\sum_{k=o}^{n}p_k\left\{\frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)}{\sin(t/2)}dt\right\}$$

$$t_n^{NE}(f;x)-f(x) = \frac{1}{2\pi P_n}\sum_{k=o}^{n}p_k\int_0^{\pi}\phi(x,t)\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\left(\frac{t}{2}\right)}{\sin(t/2)}dt$$

$$= \int_0^{\pi}\phi(x,t)(NE)_n(t)dt.$$

Using Lemma (4.3), generalized Minkowski inequality ([10, pp. 18-19]), we shall obtain the proof of this theorem in a quite different method as following:

$$\|t_n^{NE}-f\|_r = \left[\int_0^{2\pi}\left|t_n^{NE}(f;x)-f(x)\right|^r dx\right]^{\frac{1}{r}}$$

$$= \left[\int_0^{2\pi}\left|\int_0^{\pi}\phi(x,t)(NE)n(t)dt\right|^r dx\right]^{1/r}$$

$$\leq \int_0^{\pi}\left\{\int_0^{2\pi}|\phi(x,t)|^r dx\right\}^{1/r}|(NE)_n(t)|dt$$

$$= \int_0^{\pi}O(t^{\alpha})\,|(NE)_n(t)|\,dt$$

$$= O\left(\int_0^{\frac{\pi}{(n+1)}} (t^\alpha)(NE)_n(t)\,dt\right) + O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} (t^\alpha)(NE)_n(t)\,dt\right)$$
$$= I_1 + I_2. \tag{5.1}$$

Applying Lemma (4.1), we have

$$|I_1| \le O\left(\int_0^{\frac{\pi}{(n+1)}} t^\alpha (n+1)\,dt\right)$$
$$= O\left((n+1)^{-\alpha}\right). \tag{5.2}$$

Now, by Lemma (4.2), we get

$$|I_2| \le O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} \frac{t^\alpha}{(n+1)t^2}\,dt\right)$$
$$= O\left[\left(\frac{1}{(n+1)}\right)\int_{\frac{\pi}{(n+1)}}^{\pi} t^{\alpha-2}\,dt\right]$$
$$= \begin{cases} O\left[\left(\frac{1}{(n+1)}\right)\left(\frac{1}{1-\alpha}\right)\left(\frac{\pi^{\alpha-1}}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right)\right] & ,0<\alpha<1 \\ O\left(\frac{\log(n+1)}{(n+1)}\right) & , \quad \alpha = 1. \end{cases}$$
$$= \begin{cases} O\left((n+1)^{-\alpha}\right) & ,0<\alpha<1 \\ O\left(\frac{\log(n+1)}{(n+1)}\right) & , \quad \alpha = 1. \end{cases} \tag{5.3}$$

Combining equations (5.1) to (5.3), we have

$$\| t_n^{NE} - f \|_r = \begin{cases} O\left((n+1)^{-\alpha}\right) & ,0<\alpha<1 \\ O\left(\frac{\log(n+1)}{(n+1)}\right) & , \quad \alpha = 1. \end{cases}$$

This completes the proof of Theorem (1).

6. Proof of Theorem 2

Following the proof of Theorem (1), we have

$$\| t_n^{NE} - f \|_r = O\left(\int_0^{\frac{\pi}{(n+1)}} (t^\alpha)(NE)_n(t)\,dt\right) + O\left(\int_{\frac{\pi}{(n+1)}}^{\pi} (t^\alpha)(NE)_n(t)\,dt\right)$$
$$= I_1' + I_2' \quad \text{say} \tag{6.1}$$

By Lemma (4.1) and similar to proof of Theorem (1),

$$I_1' = O\left((n+1)^{-\alpha}\right). \tag{6.2}$$

Under the condition (3.2) of Theorem (2) on $\{p_n\}$, by Lemma (4.2),

$$(NE)_n(t)=\frac{1}{2t^2P_n}\sum_{k=0}^{n}\left|\Delta p_k\right|$$

Now assuming condition (3.1)(we may consider $\frac{P_n}{n+1}$ increasing in n) and take $k=k(n)$ such that $p_k=\min\{p_j:0\le j\le n\}$. Hence $(n+1)p_k\le P_n$. Now for $k\le m\le n$

$$p_m\le\sum_{j=k+1}^{m}|p_j-p_{j-1}|+|p_k|\le C\left(\frac{P_m}{m+1}\right)+\left(\frac{P_n}{n+1}\right)\le C\left(\frac{P_n}{n+1}\right)=O\left(\frac{P_n}{n+1}\right)$$

Similarly for $0\le m\le k$

$$p_m\le\sum_{j=m}^{k-1}|p_j-p_{j+1}|+|p_k|\le C\left(\frac{P_k}{k+1}\right)+\left(\frac{P_n}{n+1}\right)\le C\left(\frac{P_n}{n+1}\right)=O\left(\frac{P_n}{n+1}\right)$$

Thus the condition $\sum_{k=0}^{n}\left|\Delta p_k\right|=O\left(\frac{P_n}{(n+1)}\right)$ implies $(n+1)p_n=O(P_n)$.

Therefore $(NE)_n(t)=O\left(\frac{P_n}{(n+1)t^2}\right)$ for $\frac{\pi}{(n+1)}<t\le\pi$ and I_2' is same as I_2 of Theorem(1).

Hence, under the condition of Theorem(2) on $\{p_n\}$,

$$\|t_n^{NE}-f\|_r=\begin{cases}O\left((n+1)^{-\alpha}\right), & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right), & \alpha=1.\end{cases}$$

This completes the proof of Theorem (3.2).

7. Applications

Following corollary can be derived from Theorem (1):

Corollary 7.1. *If we take $p_n=\frac{1}{n+1}$ then the degree of approximation of a function $f\in Lip(\alpha,r)$ by $(N,\frac{1}{n+1})(E,1)$ means*

$$t_n^{HE}=\frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{1}{k+1}\frac{1}{2^{n-k}}\sum_{\nu=0}^{n-k}\binom{n-k}{k}s_\nu$$

of the series (2.1) is given by

$$\|t_n^{HE}-f\|_r=\begin{cases}O\left((n+1)^{-\alpha}\right), & 0<\alpha<1\\ O\left(\frac{\log(n+1)}{(n+1)}\right), & \alpha=1.\end{cases}$$

Corollary 7.2. *If $p_n=1\ \forall\ n\ge 1$ then degree of approximation of a function $f\in Lip(\alpha,r)$ by $(C,1)(E,1)$ means*

$$t_n^{CE}=\frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2^{n-k}}\sum_{\nu=0}^{n-k}\binom{n-k}{k}s_\nu$$

is given by

$$\| t_n^{CE} - f \|_r = \begin{cases} O\left((n+1)^{-\alpha}\right) & ,0<\alpha<1 \\ O\left(\frac{\log(n+1)}{(n+1)}\right) & , \quad \alpha = 1. \end{cases}$$

References

[1] P. Chandra, *Approximation by Norlund operators*, Mat. Vestnik, **38** (1986), 263-269.

[2] G. H. Hardy, Divergent Series, first edition, Oxford Press, p.70, 1949.

[3] H. H. Khan, *On degree of approximation of functions belonging to class* $Lip(\alpha, p)$, Indian J. Pure Appl. Math., **5**(1974), no.2, 132-136.

[4] L. McFaddin, *Absolute Nörlund summability*, Duke Math. J., 9(1942), 168-207.

[5] R. N. Mohapatra and D. C. Russell, *Some direct and inverse theorems in approximation of functions*, J. Austral. Math. Soc. (Ser. A), **34** (1983), 143-154.

[6] K. Qureshi, *On degree of approximation of a periodic function f by almost Nörlund means*, Tamkang J. Math., **12**(1981), no.1, 35-38.

[7] K. Qureshi, *On degree of approximation of functions belonging to the class* $Lip\alpha$, Indian J. Pure App. Math, **13**(1982), no.8, 898-903.

[8] B. N. Sahney and V. Rao, *Error bounds in the approximation of functions*, Bull. Austral. Math. Soc., **6**(1972), 11-18.

[9] E. C. Titchmarsh, The Theory of functions, Second Edition, Oxford University Press, 1939.

[10] A. Zygmund, Trigonometric Series, 2nd rev.ed., I, Cambridge Univ. Press, Cambridge, 1968.

Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, India.

E-mail: shyam_lal@rediffmail.com

Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, India.

E-mail: abhigal.bhu@gmail.com