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GENERALIZATIONS OF INCOMPLETE ELLIPTIC INTEGRALS OF FIRST AND SECOND KINDS

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Abstract. In this paper, we obtain analytical solutions of incomplete elliptic integrals of first and second kinds. Further, we generalize these incomplete elliptic integrals in the forms of multiple series identities involving bounded multiple sequences.

1. Introduction and Preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k=1,2,3,\cdots \\ 1 & ; \text{ if } k=0 \\ k! & ; \text{ if } b=1,\ k=1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable is defined by

$$ {}_AF_B\left[\begin{array}{c}(a_A)\,;\\ \\ (b_B)\,;\end{array} z\right]\equiv {}_AF_B\left[\begin{array}{c}(a_j)_{j=1}^{A}\,;\\ \\ (b_j)_{j=1}^{B}\,;\end{array} z\right]=\sum_{k=0}^{\infty}\frac{((a_A))_k z^k}{((b_B))_k k!} \tag{1.1}$$

Kampé de Fériet's General Double Hypergeometric Function

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H. M. Srivastava and R. Panda [13]:

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{c}(a_j)_{j=1}^{A}:(b_j)_{j=1}^{B};(d_j)_{j=1}^{D}\,;\\ \\ (e_j)_{j=1}^{E}:(g_j)_{j=1}^{G};(h_j)_{j=1}^{H}\,;\end{array} x,\ y\right]=\sum_{m,n=0}^{\infty}\frac{((a_A))_{m+n}((b_B))_m((d_D))_n\, x^m\, y^n}{((e_E))_{m+n}((g_G))_m((h_H))_n\, m!\, n!} \tag{1.2}$$

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Series Identity

It is the convention that the empty sum $\sum_{r=0}^{-1} F(r)$ is treated as zero.

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Theta(m,\ r)=\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r) \tag{1.3}$$

where $\{\Theta(m,r)\}_{m,r=0}^{\infty}$ is suitably bounded double sequence of essentially arbitrary(real or complex) parameters.

Legendre's Normal Forms of Incomplete Elliptic Integrals

Following elliptic integrals (R.H.S.) have been represented in different notations (L.H.S.) by researchers

First Kind : $$F(x,\ \phi)=\int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sin^2\theta)}} \tag{1.4}$$

Second Kind : $$E(x,\ \phi)=\int_0^{\phi}\sqrt{(1-x^2\sin^2\theta)}\ \mathrm{d}\theta \tag{1.5}$$

where $\quad 0\le\ x\le\ 1,\ 0\le\ \phi\le\ \frac{\pi}{2}$

The integrands of elliptic integrals are periodic functions with a period π. Here x and ϕ are called modulus and amplitude respectively. If $x=\sin\delta$, then δ is called modular angle.

Some Useful Indefinite Integrals

When $m=0,1,2,3,\cdots,$ then

$$\int\sin^{2m}(c\theta)\mathrm{d}\theta=\left\{\frac{-(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\sin^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+\text{Constant} \tag{1.6}$$

$$\int\cos^{2m}(c\theta)\mathrm{d}\theta=\left\{\frac{(\frac{1}{2})_m\sin(c\theta)\cos(c\theta)}{(1)_m\ c}\sum_{r=0}^{m-1}\frac{(1)_r\cos^{2r}(c\theta)}{(\frac{3}{2})_r}\right\}+\left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\}+\text{Constant} \tag{1.7}$$

$$\int\sin^{2m+1}(c\theta)\mathrm{d}\theta=\frac{-(1)_m\cos(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\sin^{2r}(c\theta)}{(1)_r}+\text{Constant} \tag{1.8}$$

$$\int\cos^{2m+1}(c\theta)\mathrm{d}\theta=\frac{(1)_m\sin(c\theta)}{(\frac{3}{2})_m\ c}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\cos^{2r}(c\theta)}{(1)_r}+\text{Constant} \tag{1.9}$$

2. A general family of multiple-series identities

Theorem 1. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\Big(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\Big)\frac{y^m}{m!}$$

$$= -\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \quad (2.1)$$

provided that each of the series involved is absolutely convergent.

Theorem 2. *Let* $\{\Omega_m\}_{m=0}^{\infty}$ *be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\Big(\int_0^{\gamma}\cos^{2m}(c\theta)\mathrm{d}\theta\Big)\frac{y^m}{m!}$$
$$=\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ y^m(y\cos^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \quad (2.2)$$

provided that each of the series involved is absolutely convergent.

Theorem 3. *Let* $\{\Omega_m\}_{m=0}^{\infty}$ *be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\Big(\int_0^{\gamma}\sin^{2m+1}(c\theta)\mathrm{d}\theta\Big)\frac{y^m}{m!}=\frac{1}{c}\sum_{m=0}^{\infty}\Omega_m\frac{y^m}{(\frac{3}{2})_m}-\frac{\cos(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\ y^m(y\sin^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r} \quad (2.3)$$

provided that each of the series involved is absolutely convergent.

Theorem 4. *Let* $\{\Omega_m\}_{m=0}^{\infty}$ *be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\Big(\int_0^{\gamma}\cos^{2m+1}(c\theta)\mathrm{d}\theta\Big)\frac{y^m}{m!}=\frac{\sin(c\gamma)}{c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\ y^m(y\cos^2(c\gamma))^r}{(\frac{3}{2})_{m+r}(1)_r} \quad (2.4)$$

provided that each of the series involved is absolutely convergent.

Proof of (2.1)**.**

$$\sum_{m=0}^{\infty}\Omega_m\Big(\int_0^{\gamma}\sin^{2m}(c\theta)\mathrm{d}\theta\Big)\frac{y^m}{m!}$$
$$=-\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Omega_m\frac{\sin(c\gamma)\cos(c\gamma)(\frac{1}{2})_m(1)_r\ y^m(\sin^2(c\gamma))^r}{c(1)_m(1)_m(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2}$$
$$=-\frac{y\sin(c\gamma)\cos(c\gamma)}{2c}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ y^m(y\sin^2(c\gamma))^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}+\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2}$$

Similarly we can derive (2.2) to (2.4).

3. Hypergeometric generalizations

Putting $c=1$ in theorems (2.1) to (2.4) and setting $\Omega_m=\frac{(a_1)_m(a_2)_m(a_3)_m\cdots(a_A)_m}{(b_1)_m(b_2)_m(b_3)_m\cdots(b_B)_m}=\frac{((a_A))_m}{((b_B))_m}$, using some algebraic properties of Pochhammer symbol and interpreting the multiple power series in hypergeometric notations given by (1.1)-(1.2), we get the analytical solutions of generalized incomplete elliptic integrals.

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\,;\\ \\ (b_j)_{j=1}^B\,;\end{matrix}\; y\sin^2\theta\right]\mathrm{d}\theta=\gamma\,{}_{A+1}F_{B+1}\left[\begin{matrix}\frac{1}{2},(a_j)_{j=1}^A\,;\\ \\ 1,(b_j)_{j=1}^B\,;\end{matrix}\; y\right]$$
$$-\frac{y\sin\gamma\cos\gamma\prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)}F^{A+1:1;2}_{B+2:0;1}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^A\,:1\,;1,1\,;\\ \\ 2,2,(1+b_j)_{j=1}^B:-;\frac{3}{2}\,;\end{matrix}\; y,y\sin^2\gamma\right] \tag{3.1}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\,;\\ \\ (b_j)_{j=1}^B\,;\end{matrix}\; y\cos^2\theta\right]\mathrm{d}\theta=\gamma\,{}_{A+1}F_{B+1}\left[\begin{matrix}\frac{1}{2},(a_j)_{j=1}^A\,;\\ \\ 1,(b_j)_{j=1}^B\,;\end{matrix}\; y\right]$$
$$+\frac{y\sin\gamma\cos\gamma\prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)}F^{A+1:1;2}_{B+2:0;1}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^A\,:1\,;1\,,\,1\,;\\ \\ 2,2,(1+b_j)_{j=1}^B:-;\,\frac{3}{2}\,;\end{matrix}\; y,y\cos^2\gamma\right] \tag{3.2}$$

$$\int_0^{\gamma}\sin\theta\,{}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\,;\\ \\ (b_j)_{j=1}^B\,;\end{matrix}\; y\sin^2\theta\right]\mathrm{d}\theta={}_{A+1}F_{B+1}\left[\begin{matrix}1,(a_j)_{j=1}^A\,;\\ \\ \frac{3}{2},(b_j)_{j=1}^B\,;\end{matrix}\; y\right]$$
$$-\cos\gamma\,F^{A\;:1;1}_{B+1:0;0}\left[\begin{matrix}(a_j)_{j=1}^A\,:1\,;\frac{1}{2}\,;\\ \\ \frac{3}{2},(b_j)_{j=1}^B:-;-\,;\end{matrix}\; y,\;y\sin^2\gamma\right] \tag{3.3}$$

$$\int_0^{\gamma}\cos\theta\,{}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\,;\\ \\ (b_j)_{j=1}^B\,;\end{matrix}\; y\cos^2\theta\right]\mathrm{d}\theta=\sin\gamma\,F^{A\;:1;1}_{B+1:0;0}\left[\begin{matrix}(a_j)_{j=1}^A\,:1\,;\,\frac{1}{2}\,;\\ \\ \frac{3}{2},(b_j)_{j=1}^B:-;-\,;\end{matrix}\; y,y\cos^2\gamma\right] \tag{3.4}$$

provided that each of the series as well as associated integrals involved are convergent.

4. Solutions of incomplete elliptic integrals

Setting $A=1,B=0$ and $a_1=\frac{1}{2}$ in (3.1) and (3.2) respectively, we get

$$F(\sqrt{y},\gamma)=\int_0^{\gamma}\frac{\mathrm{d}\theta}{\sqrt{(1-y\sin^2\theta)}}=\gamma\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\,;\\ \\ 1\;\;\;;\end{matrix}\; y\right]-\frac{y\sin\gamma\cos\gamma}{4}$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{3}{2},\frac{3}{2}: 1\,; 1,1\,; \\ & y, y\sin^2\gamma \\ 2,2:\text{—};\ \frac{3}{2}\quad ; \end{matrix}\right] \quad ;|y|<1 \tag{4.1}$$

which is the exact solution of incomplete elliptic integral of first kind.

$$\int_0^{\gamma}\frac{d\theta}{\sqrt{(1-y\cos^2\theta)}} = \gamma\, {}_2F_1\left[\begin{matrix} \frac{1}{2},\frac{1}{2}\,; \\ & y \\ 1\quad ; \end{matrix}\right] + \frac{y\sin\gamma\cos\gamma}{4} \times F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{3}{2},\frac{3}{2}: 1; 1,1\,; \\ & y, y\cos^2\gamma \\ 2,2:\text{—};\ \frac{3}{2}\quad ; \end{matrix}\right] \quad ;|y|<1 \tag{4.2}$$

Putting $A=1, B=0$ and $a_1=-\frac{1}{2}$ in (3.1) and (3.2)respectively, we get

$$E(\sqrt{y},\gamma) = \int_0^{\gamma}\sqrt{(1-y\sin^2\theta)}\,d\theta = \gamma\, {}_2F_1\left[\begin{matrix} \frac{1}{2},-\frac{1}{2}\,; \\ & y \\ 1\quad ; \end{matrix}\right] + \frac{y\sin\gamma\cos\gamma}{4} \times F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{1}{2},\frac{3}{2}: 1; 1,1\,; \\ & y, y\sin^2\gamma \\ 2,2:\text{—};\ \frac{3}{2}\quad ; \end{matrix}\right] \quad ;|y|<1 \tag{4.3}$$

which is the exact solution of incomplete elliptic integral of second kind.

$$\int_0^{\gamma}\sqrt{(1-y\cos^2\theta)}\,d\theta = \gamma\, {}_2F_1\left[\begin{matrix} \frac{1}{2},-\frac{1}{2}\,; \\ & y \\ 1\quad ; \end{matrix}\right] - \frac{y\sin\gamma\cos\gamma}{4} \times F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{1}{2},\frac{3}{2}:1\,; 1,1\,; \\ & y, y\cos^2\gamma \\ 2,2:\text{—};\ \frac{3}{2}\quad ; \end{matrix}\right] \quad ;|y|<1 \tag{4.4}$$

These solutions are not found in notebooks [1, 2, 3, 4, 10, 11, 12, 13] and other literature [5, 6, 7, 8, 9].

5. Special cases

When $\gamma = \frac{\pi}{2}$ in (4.1) and (4.3) we get well known solutions related with complete elliptic integrals of first and second kinds respectively.

In (3.1), put $A=2, B=1, a_1=b, a_2=-b, b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we get

$$\int_0^{\frac{\pi}{2}} \cos\{2b\sin^{-1}(\sqrt{y}\sin\theta)\}\,\mathrm{d}\theta = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix} b, -b\,; \\ \\ 1 \quad ; \end{matrix} \; y\right] \tag{5.1}$$

In (3.1), set $A=2, B=1, a_1=b, a_2=1-b, b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we obtain a known result of Ramanujan [3, p.88(Entry 1)].

In (3.1), substitute $A=2, B=1, a_1=b, a_2=1-b, b_1=\frac{3}{2}$ and $\gamma=\frac{\pi}{2}$, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{\sin\{(2b-1)\sin^{-1}(\sqrt{y}\sin\theta)\}}{\sin\theta}\,\mathrm{d}\theta = \frac{\pi(2b-1)\sqrt{y}}{2}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}, b, 1-b\,; \\ \\ 1, \frac{3}{2} \quad ; \end{matrix} \; y\right] \tag{5.2}$$

In (3.1), put $A=2, B=1, a_1=b, a_2=2-b, b_1=\frac{3}{2}$ and $\gamma=\frac{\pi}{2}$, we get

$$\int_0^{\frac{\pi}{2}} \frac{\sin\{(2b-2)\sin^{-1}(\sqrt{y}\sin\theta)\}}{\sin\theta\sqrt{(1-y\sin^2\theta)}}\,\mathrm{d}\theta = \pi(b-1)\sqrt{y}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}, b, 2-b\,; \\ \\ 1, \frac{3}{2} \quad ; \end{matrix} \; y\right] \tag{5.3}$$

In (3.1), set $A=2, B=1, a_1=b, a_2=b+\frac{1}{2}, b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we get

$$\int_0^{\frac{\pi}{2}} \frac{\cos\{2b\tan^{-1}(\sqrt{y}\sin\theta)\}}{(1+y\sin^2\theta)^b}\,\mathrm{d}\theta = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix} b, b+\frac{1}{2}\,; \\ \\ 1 \quad ; \end{matrix} \; -y\right] \tag{5.4}$$

In (3.1), substitute $A=2, B=1, a_1=b, a_2=b+\frac{1}{2}, b_1=\frac{3}{2}$ and $\gamma=\frac{\pi}{2}$, we can find

$$\int_0^{\frac{\pi}{2}} \frac{(1+y\sin^2\theta)^{\frac{1}{2}-b}\sin\{(2b-1)\tan^{-1}(\sqrt{y}\sin\theta)\}}{\sin\theta}\,\mathrm{d}\theta$$
$$= \frac{\pi(2b-1)\sqrt{y}}{2}\,{}_3F_2\left[\begin{matrix} \frac{1}{2}, b, b+\frac{1}{2}\,; \\ \\ 1, \frac{3}{2} \quad ; \end{matrix} \; -y\right] \tag{5.5}$$

In (3.1), put $A=2, B=1, a_1=\frac{1}{3}, a_2=\frac{2}{3}, b_1=\frac{1}{2}$ and $\gamma=\frac{\pi}{2}$, we obtain a known result of B. C. Berndt [4, p.133].

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