TAMKANG JOURNAL OF MATHEMATICS
Volume 43, Number 3, 417-421, Autumn 2012
doi:10.5556/j.tkjm.43.2012.417-421
Available online at http://journals.math.tku.edu.tw/

DIFFERENTIAL OPERATOR OF MEROMORPHIC p-VALENT FUNCTIONS

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Abstract. A certain differential operator $I_\lambda^m f(z)$ is introduced for functions of the form $f(z)=\frac{1}{z^p}+\sum_{n=k}^{\infty} a_{n+p-1}z^{n+p-1}$ which are p-valent in the punctured unit disk $\mathcal{U}^*=\{z : z\in\mathbb{C}$ and $0<|z|<1\}$, where p and k are positive integers. The main object of this paper is to give an application of the operator $I_\lambda^m f(z)$ to the differential inequalities.

1. Introduction and definitions

Let $\sum_{p,k}$ denote the class of functions of the form :

$$f(z)=\frac{1}{z^p}+\sum_{n=k}^{\infty} a_{n+p-1}z^{n+p-1}, \qquad (p,k\in\mathbb{N}:=\{1,2,3,\ldots\}), \tag{1.1}$$

which are p-valent in the punctured unit disk $\mathcal{U}^*=\{z : z\in\mathbb{C}$ and $0<|z|<1\}$. For a function f in $\sum_{p,k}$, we define the following new differential operator:

$$\begin{aligned} I^0 f(z) &= f(z),\\ I_\lambda^1 f(z) &= (1-\lambda)f(z)+\lambda z f'(z)+\frac{\lambda(p+1)}{z^p}, \quad \lambda\ge 0,\\ I_\lambda^2 f(z) &= (1-\lambda)I_\lambda^1 f(z)+\lambda z(I_\lambda^1 f(z))'+\frac{\lambda(p+1)}{z^p}, \end{aligned}$$

and for $m=1,2,3,\ldots$

$$\begin{aligned} I_\lambda^m f(z) &= (1-\lambda)I_\lambda^{m-1} f(z)+\lambda z(I_\lambda^{m-1} f(z))'+\frac{\lambda(p+1)}{z^p}\\ &= \frac{1}{z^p}+\sum_{n=k}^{\infty}[1+\lambda(p+n-2)]^m a_{n+p-1}z^{n+p-1} \end{aligned} \tag{1.2}$$

Note that for $\lambda=p=k=1$, we have the operator $I^m f(z)$ introduced and studied by Frasin and Darus [2].

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic functions, meromorphic p-valent functions.

It easily verified from (1.2) that

$$\lambda z(I_\lambda^m f(z))' = I_\lambda^{m+1} f(z) - (1-\lambda) I_\lambda^m f(z) - \frac{\lambda(p+1)}{z^p}. \tag{1.3}$$

From the identity (1.3), we readily have

$$\lambda z(I_\lambda^{m-1} f(z))' = I_\lambda^{m} f(z) - (1-\lambda) I_\lambda^{m-1} f(z) - \frac{\lambda(p+1)}{z^p} \tag{1.4}$$

and

$$\lambda z(I_\lambda^{m+1} f(z))' = I_\lambda^{m+2} f(z) - (1-\lambda) I_\lambda^{m+1} f(z) - \frac{\lambda(p+1)}{z^p}. \tag{1.5}$$

Very recently, Kamali [3] has obtained new properties of meromorphic p-valent functions defined by Ruscheweyh operator $D^{n+p-1} f(z)$ [6] (see also [7, 1, 4]).

The object of the present paper is to investigate some new properties of meromorphic p-valent functions by the above operator $I_\lambda^m f(z)$ given by (1.2).

Definition 1.1. *Let H be the set of complex valued functions* $h(r,s,t):\mathbb{C}^3 \to \mathbb{C}$ such that

$$h(r,s,t) \text{ is continuous in a domain } \mathbb{D} \subset \mathbb{C}^3; \tag{1.6}$$

$$(1,1,1) \in \mathbb{D} \quad \text{and} \quad |h(1,1,1)| < 1; \tag{1.7}$$

$$\left| h\left(e^{i\theta}, e^{i\theta} + \lambda\delta, \frac{\lambda^2(\delta+\beta) + 3\lambda\delta e^{i\theta} + e^{2i\theta}}{e^{i\theta} + \lambda\delta}\right)\right| \geq 1 \tag{1.8}$$

whenever

$$\left(e^{i\theta}, e^{i\theta} + \lambda\delta, \frac{\lambda^2(\delta+\beta) + 3\lambda\delta e^{i\theta} + e^{2i\theta}}{e^{i\theta} + \lambda\delta}\right) \in \mathbb{D}$$

with $\mathrm{Re}\,\beta \geq \delta(\delta-1)$ *for real* θ, $\delta \geq 1$ *and* $\lambda > 0$.

2. Main result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu [5].

Lemma 2.1. *Let $w(z) = a + w_n z^n + \cdots$, be analytic in $\mathcal{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$ with $w(z) \neq a$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z|\leq r_0} |w(z)|$. Then*

$$zw'(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$\mathrm{Re}\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \delta, \tag{2.2}$$

where δ is a real number and

$$\delta \geq n\frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq n\frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Theorem 2.2. *Let $h(r,s,t) \in H$ and let $f(z) \in \sum_{p,k}$ satisfies*

$$\left(\frac{I_\lambda^m f(z)}{I_\lambda^{m-1} f(z)}, \frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}, \frac{I_\lambda^{m+2} f(z)}{I_\lambda^{m+1} f(z)}\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.3}$$

and

$$\left|h\left(\frac{I_\lambda^m f(z)}{I_\lambda^{m-1} f(z)}, \frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)}, \frac{I_\lambda^{m+2} f(z)}{I_\lambda^{m+1} f(z)}\right)\right| < 1 \tag{2.4}$$

for all $z \in \mathcal{U}$ and for some $m \in \mathbb{N}$. Then we have

$$\left|\frac{I_\lambda^m f(z)}{I_\lambda^{m-1} f(z)}\right| < 1 \qquad (z \in \mathcal{U}; \lambda > 0).$$

Proof. Let

$$\frac{I_\lambda^m f(z)}{I_\lambda^{m-1} f(z)} = w(z). \tag{2.5}$$

Then it follows that $w(z)$ is either analytic or meromorphic in $\mathcal{U}$, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.5) logarithmically and multiply by z, we obtain

$$\frac{z(I_\lambda^m f(z))'}{I_\lambda^m f(z)} - \frac{z(I_\lambda^{m-1} f(z))'}{I_\lambda^{m-1} f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.3) and (1.4), we have

$$\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)} = w(z) + \lambda \frac{zw'(z)}{w(z)}. \tag{2.6}$$

Differentiating (2.6) logarithmically and multiply by z, we have

$$\begin{aligned}\frac{z(I_\lambda^{m+1} f(z))'}{I_\lambda^{m+1} f(z)} - \frac{z(I_\lambda^m f(z))'}{I_\lambda^m f(z)} &= \frac{z\left[w(z) + \lambda \frac{zw'(z)}{w(z)}\right]'}{w(z) + \lambda \frac{zw'(z)}{w(z)}} \\ &= \frac{zw'(z) + \lambda\left[\frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z) + \lambda \frac{zw'(z)}{w(z)}}.\end{aligned} \tag{2.7}$$

Using the identities (1.3) and (1.5), it follows from (2.7) that

$$\begin{aligned}\frac{1}{\lambda}\frac{I_\lambda^{m+2} f(z)}{I_\lambda^{m+1} f(z)} &= \frac{1}{\lambda}\frac{I_\lambda^{m+1} f(z)}{I_\lambda^m f(z)} + \frac{zw'(z) + \lambda\left[\frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z) + \lambda \frac{zw'(z)}{w(z)}} \\ &= \frac{1}{\lambda} w(z) + \frac{zw'(z)}{w(z)} + \frac{zw'(z) + \lambda\left[\frac{zw'(z)}{w(z)} + \frac{z^2 w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z) + \lambda \frac{zw'(z)}{w(z)}}.\end{aligned}$$

We claim that $|w(z)| < 1$ for $z \in \mathcal{U}$. Otherwise there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z|\le r_0} |w(z)| = |w(z_0)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 with $a = 1$ and $n = 1$, we have

$$\frac{I_\lambda^m f(z_0)}{I_\lambda^{m-1} f(z_0)} = e^{i\theta},$$

$$\frac{I_\lambda^{m+1} f(z_0)}{I_\lambda^{m} f(z_0)} = e^{i\theta} + \lambda\delta,$$

$$\frac{I_\lambda^{m+2} f(z_0)}{I_\lambda^{m+1} f(z_0)} = \frac{\lambda^2(\delta+\beta) + 3\lambda\delta e^{i\theta} + e^{2i\theta}}{e^{i\theta} + \lambda\delta},$$

where

$$\beta = \frac{z_0^2 w''(z_0)}{w(z_0)} \qquad \text{and} \quad \delta \ge 1.$$

Further, an application of (2.2) in Lemma 2.1 gives $\mathrm{Re}\beta \ge \delta(\delta-1)$.Since $h(r,s,t) \in H$, we have

$$\begin{aligned} &\left| h\left(\frac{I_\lambda^m f(z_0)}{I_\lambda^{m-1} f(z_0)}, \frac{I_\lambda^{m+1} f(z_0)}{I_\lambda^{m} f(z_0)}, \frac{I_\lambda^{m+2} f(z_0)}{I_\lambda^{m+1} f(z_0)} \right) \right| \\ &= \left| h\left(e^{i\theta}, e^{i\theta} + \lambda\delta, \frac{\lambda^2(\delta+\beta) + 3\lambda\delta e^{i\theta} + e^{2i\theta}}{e^{i\theta} + \lambda\delta} \right) \right| \\ &\ge 1 \end{aligned}$$

which contradicts the condition (2.4) of Theorem 2.2. Therefore, we conclude that

$$\left| \frac{I_\lambda^m f(z)}{I_\lambda^{m-1} f(z)} \right| < 1 \qquad (z \in \mathcal{U}).$$

The proof is complete. □

Letting $\lambda = 1$ and $m = 1$ in Theorem 2.2, we have

Corollary 2.3. *Let $h(r,s,t) \in H$ and let $f(z) \in \sum_{p,k}$ satisfies*

$$\left(\frac{z^{p+1}f'(z)+p+1}{z^p f(z)}, \frac{z^{p+2}f''(z)+z^{p+1}f'(z)+1-p^2}{z^{p+1}f'(z)+p+1}, \frac{z^{p+3}f'''(z)+3z^{p+2}f''(z)+z^{p+1}f'(z)+1+p^3}{z^{p+2}f''(z)+z^{p+1}f'(z)+1-p^2} \right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.8}$$

and

$$\left| h\left(\frac{z^{p+1}f'(z)+p+1}{z^p f(z)}, \frac{z^{p+2}f''(z)+z^{p+1}f'(z)+1-p^2}{z^{p+1}f'(z)+p+1}, \frac{z^{p+3}f'''(z)+3z^{p+2}f''(z)+z^{p+1}f'(z)+1+p^3}{z^{p+2}f''(z)+z^{p+1}f'(z)+1-p^2} \right) \right| < 1 \tag{2.9}$$

for all $z \in \mathcal{U}$. Then we have

$$\left| \frac{z^{p+1}f'(z)+p+1}{z^p f(z)} \right| < 1 \qquad (z \in \mathcal{U}).$$

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