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GENERALIZED VANDERMONDE DETERMINANTS FOR REVERSING TAYLOR'S FORMULA AND APPLICATION TO HYPOELLIPTICITY

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Abstract. The problem of the hypoellipticity of the linear partial differential operators with constant coefficients was completely solved by Hörmander in [5]. He listed many equivalent algebraic conditions on the polynomial symbol of the operator, each necessary and sufficient for hypoellipticity. In this paper we employ two Mitchell's Theorems (1881) regarding a type of Generalized Vandermonde Determinants, for inverting Taylor's formula of polynomials in several variables with complex coefficients. We obtain then a more direct and easy proof of an equivalence for the mentioned Hörmander's hypoellipticity conditions.

1. Notations and basic definitions

We begin by presenting some standard notations and basic definitions of the general theory of the linear partial differential equations, in according to Boggiatto-Buzano-Rodino [3], Trèves [8], for instance.

Let n be an integer, $n \geq 1$. By $\mathbb{Z}_+^n$ we denote the subset of $\mathbb{R}^n$ consisting of elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ whose components α_j are non negative integers; these elements α will be called multi-indices. The set $\mathbb{Z}_+^n$ is turned into a partially ordered set by the order relation $\alpha \leq \beta$, meaning that $\alpha_j \leq \beta_j$ for all $j = 1, \ldots, n$; $\alpha < \beta$ means $\alpha \leq \beta$ but there exists j with $\alpha_j < \beta_j$. Let $x = (x_1, \ldots, x_n)$ be the variables in $\mathbb{R}^n$ and let $\xi = (\xi_1, \ldots, \xi_n)$ be the dual variables of x, we shall also make a systematic use of the following notations:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \ , \ \alpha! = \alpha_1! \cdots \alpha_n! \ , \ \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \ , \ x\xi = x_1\xi_1 + \cdots + x_n\xi_n.$$

We write $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$; using the notation $D_{x_j} = -i\frac{\partial}{\partial x_j}$ where i is the imaginary unit, we also write $D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$.

Fixed a non negative integer m, a linear partial differential operator P of order m with constant coefficients in $\mathbb{R}^n$ is defined as follows:

$$P = P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha \quad , \quad c_\alpha = c_{\alpha_1, \ldots, \alpha_n} \in \mathbb{C} \ , \tag{1.1}$$

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where $\mathbb{C}$ represents the set of the complex numbers.

The polynomial

$$P(\xi) = e^{-ix\xi}P(e^{ix\xi}) = e^{-ix\xi} \sum_{|\alpha|\le m} c_\alpha (-i)^{|\alpha|}\partial^\alpha (e^{ix\xi}) = \sum_{|\alpha|\le m} c_\alpha \xi^\alpha \tag{1.2}$$

associated to the operator $P(D)$ in (1.1) is called the full symbol of $P(D)$. There exists a one-to-one correspondence between the class of differential operators with constant coefficients $P(D)$ in (1.1) and the class of polynomials with complex coefficients $P(\xi)$ in (1.2). The polynomials of degree m, in n variables, with complex coefficients, form a linear space over the complex space, whose dimension we shall denote by $N(m,n)$. An easy computation shows that $N(m,n) = \frac{(m+n)!}{m!n!}$. This corresponds, of course, to the number of the coefficients c_α or equivalently of the multi-indices α in (1.2).
With $P^\alpha(\xi)$ we denote the polynomial $\partial^\alpha P(\xi)$, and for $\theta \in \mathbb{R}^n$, the expression

$$P(\xi+\theta) = \sum_\alpha \frac{P^\alpha(\xi)}{\alpha!}\theta^\alpha \ ,$$

represents Taylor's formula for $P(\xi)$.

The operator P in (1.1) is said to be hypoelliptic (or C^∞-regular) in Ω, open subset of $\mathbb{R}^n$, if all $u \in D'(\Omega)$ solutions of $P(D)u = 0$ shall be in $C^\infty(\Omega)$.

2. Introduction and statement of the result

Several algebraic conditions on the full symbol $P(\xi)$ in (1.2), each necessary and sufficient in order that the corresponding operator $P(D)$ is hypoelliptic, are listed in Hörmander [5]. Two of these conditions are the following:

a) $\frac{P^\alpha(\xi)}{P(\xi)} \to 0$ as $|\xi| \to +\infty$ for every $\alpha \neq 0$;

b) For every fixed $\theta \in \mathbb{R}^n$ $\frac{P(\xi+\theta)}{P(\xi)} \to 1$ as $|\xi| \to +\infty$.

That the condition b) is necessary for a) it follows immediately by applying Taylor's formula to $P(\xi+\theta)$, whereas the sufficiency of b) is proved in Hörmander [5, Lemma 2.10] by a delicate argument of "algebraic genericity". Besides, Hörmander [6, p.200, lines 4-9] actually appeals to the implication b) $\to$ a) to observe that in a) first-order derivatives suffice. Namely a) and b) are equivalent to

c) $\frac{P^{e^j}(\xi)}{P(\xi)} \to 0$ as $|\xi| \to +\infty$, with e^j denoting the $\mathrm{j-th}$ base unit vector.

Condition c) gives considerable simplifications to verify that a polynomial satisfies the condition a), and so, that its corresponding operator is hypoelliptic too.

For others equivalent conditions of hypoellipticity we address to Hörmander [7, Chapter 11, Section 1].

In the present paper we propose a simple and alternative proof for the Hörmander's equivalences of the conditions a), b) and c). In particular for showing that b) is also a sufficient condition for a) we resort to the so called Generalized Vandermonde Determinants (GVD) studied in Mitchell [4] . Two Mitchell's Theorems about GVD allow us to converse Taylor's formula for polynomials in $\mathbb{R}^n$ with complex coefficients, namely allow us to write each α-derivative $P^{\alpha}(\xi)$ of $P(\xi)$ as well as a linear combination of translations of $P(\xi)$:

$$P^{\alpha}(\xi) = \sum_{k=1}^{N} t_k P(\xi + \theta_k) \ , \tag{2.1}$$

where $N = N(m,n)$, $\theta_k \in \mathbb{R}^n$ and $t_k = t_k(\alpha) \in \mathbb{R}$.

Let now $a = (a_1, \ldots, a_N)$ be a fixed N-tuple of non negative integers and let $GVD(\eta_1, \ldots, \eta_N) = GVD$ be the polynomial obtained by computing the determinant of the $N \times N$ Generalized Vandermonde matrix

$$\begin{pmatrix} \eta_1^{a_1} & \eta_2^{a_1} & \cdots & \eta_N^{a_1} \\ \eta_1^{a_2} & \eta_2^{a_2} & \cdots & \eta_N^{a_2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \eta_1^{a_N} & \eta_2^{a_N} & \cdots & \eta_N^{a_N} \end{pmatrix} .$$

Under the hypothesis that $0 \le a_1 < a_2 < \cdots < a_N$, O.H. Mitchell [4] proved that the Generalized Vandermonde Determinant GVD of the previous matrix is:

$$GVD = \prod_{1 \le i < j \le N} (\eta_j - \eta_i) \, \mathrm{Pol}(\eta_1, \ldots, \eta_N) \ ,$$

where $\mathrm{Pol}(\eta_1, \ldots, \eta_N)$ is a polynomial in the indeterminates η_k, $k = 1, \ldots, N$.

Under the same hypothesis, Mitchell proved in [4] that the coefficients of $\mathrm{Pol}(\eta_1, \ldots, \eta_N)$ are non negative and that the sum S of them is:

$$S = \frac{\prod_{1 \le i < j \le N} (a_j - a_i)}{\prod_{1 \le h \le N-1} (N-h)!} \ > 0 \ .$$

For related easy proofs see also Evans-Isaacs [2].

In the next Section 3 we prove formula (2.1), see Theorem 3.1. Finally we easily deduce the complete proof of the equivalences of the conditions a), b) and c), see Theorem 3.2. Again we emphasize that the final result is not new, cf. [5], [6], but because of the importance of condition c) in the applications, we hope our elementary proof may have some other use.

3. Inversion of Taylor's formula for polynomials and hypoellipticity

Theorem 3.1. *Let $P(\xi)=\sum_{|\alpha|\le m} c_\alpha \xi^\alpha$, $\xi\in\mathbb{R}^n$, be a polynomial of degree $m\ge 0$ and let $T=\{\theta_k\}_{k=1,\dots,N}$, with $N=N(m,n)=\frac{(m+n)!}{m!n!}$, be a set of elements $\theta_k=(\theta_{k,1},\dots,\theta_{k,n})\in\mathbb{R}^n$ with $\theta_{k,j}>0$, $j=1,\dots,n$, such that:*

$$\begin{cases}\theta_{k,2}=\theta_{k,1}^{m+1}\\ \theta_{k,3}=\theta_{k,1}^{m^2+m+1}\\ \theta_{k,4}=\theta_{k,1}^{m^3+m^2+m+1}\\ \quad\cdot \qquad \cdot\\ \quad\cdot \qquad \cdot\\ \quad\cdot \qquad \cdot\\ \theta_{k,n}=\theta_{k,1}^{m^{n-1}+\cdots+m+1}\end{cases} \qquad k=1,\dots,N\ , \tag{3.1}$$

and

$$\theta_{i,1}\neq\theta_{j,1}\ ,\ \text{for all } i\neq j\ ,\ i,j=1,\dots,N\ . \tag{3.2}$$

Then, for all $\alpha\in\mathbb{Z}_+^n$ there exists a set $\{t_k\}_{k=1,\dots,N}$, $t_k\in\mathbb{R}$, such that

$$P^\alpha(\xi)=\sum_{k=1}^N t_k P(\xi+\theta_k)\ . \tag{3.3}$$

If $\alpha\neq 0$ then $\sum_{k=1}^N t_k=0$.

Proof. For all $k=1,\dots,N$ we consider Taylor's formula for $P(\xi+\theta_k)$:

$$P(\xi+\theta_k)=P(\xi)+\sum_{|\alpha|\neq 0}\frac{P^\alpha(\xi)}{\alpha!}\theta_k^\alpha\ ,$$

and multiply the left-hand and right-hand sides for a real variable t_k. Summing on k from 1 to N in both sides we obtain:

$$\sum_{k=1}^N t_k P(\xi+\theta_k)=P(\xi)\sum_{k=1}^N t_k+\sum_{|\alpha|\neq 0}\frac{P^\alpha(\xi)}{\alpha!}\sum_{k=1}^N t_k\theta_k^\alpha\ . \tag{3.4}$$

From hypotheses (3.1) we have that

$$\theta_k^\alpha=\theta_{k,1}^{p(\alpha)} \tag{3.5}$$

with

$$p(\alpha)=\alpha_1+\alpha_2(m+1)+\cdots+\alpha_n(m^{n-1}+\cdots+m+1). \tag{3.6}$$

By formulas (3.4) and (3.5) it follows that for all $\alpha\in\mathbb{Z}_+^n\setminus\{0\}$, $P^\alpha(\xi)$ can be written in the form (3.3) if the indeterminates t_k, $k=1,\dots,N$ solve the following inhomogeneous linear system

$$\begin{cases} \sum_{k=1}^{N} t_k & = 0 \\ \sum_{k=1}^{N} t_k \, \eta_k^{p(\alpha)} & = \alpha! \quad , \\ \forall \, \beta \neq \alpha \; , \; \sum_{k=1}^{N} t_k \, \eta_k^{p(\beta)} & = 0 \end{cases} \tag{3.7}$$

where we set $\mathbb{R}_+ \setminus \{0\} \ni \eta_k = \theta_{k,1}$ for all $k = 1, \ldots, N$. Since the number of multi-indices α with $|\alpha| \leq m$ is exactly $\frac{(m+n)!}{m!n!}$, (3.7) also consists of $N(m,n)$ equations, and the associated matrix is the following Generalized Vandermonde one:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \eta_1^{p(\alpha)} & \eta_2^{p(\alpha)} & \eta_3^{p(\alpha)} & \cdots & \eta_N^{p(\alpha)} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \eta_1^{p(\beta)} & \eta_2^{p(\beta)} & \eta_3^{p(\beta)} & \cdots & \eta_N^{p(\beta)} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \end{pmatrix} .$$

In view of Mitchell's Theorems if $p(\alpha) \neq p(\beta)$ for $\alpha \neq \beta$ we can conclude that

$$\mathrm{GVD} = \prod_{1 \leq i < j \leq n} (\eta_j - \eta_i) \mathrm{Pol}(\eta_1, \ldots, \eta_N) \neq 0 \; ,$$

because of the hypotheses (3.2) and since $\eta_k > 0$, $k = 1, \ldots, N$, implies $\mathrm{Pol}(\eta_1, \ldots, \eta_N) > 0$. So, the system (3.7) admits unique solution. For $\alpha = 0$, $P^0(\xi) = P(\xi)$ is obtained by putting $\sum_{k=1}^{N} t_k = 1$ and for all $\beta \neq 0$, $\sum_{k=1}^{N} t_k \, \eta_k^{p(\beta)} = 0$ in the system (3.7).

Now it remains to be proved that $p(\alpha) = p(\beta)$ implies $\alpha = \beta$. Let

$$\begin{array}{c} \alpha_1 + \alpha_2(m+1) + \cdots + \alpha_n(m^{n-1} + \cdots + m + 1) = \\ \beta_1 + \beta_2(m+1) + \cdots + \beta_n(m^{n-1} + \cdots + m + 1) \end{array} , \tag{3.8}$$

and define d_1, e_1 by

$$1 \leq \alpha_1 + \cdots + \alpha_n = d_1 \leq m \quad \text{and} \quad 1 \leq \beta_1 + \cdots + \beta_n = e_1 \leq m \tag{3.9}$$

From (3.9) we have

$$\alpha_1 = d_1 - \alpha_2 - \ldots - \alpha_n \quad \text{and} \quad \beta_1 = e_1 - \beta_2 - \ldots - \beta_n \; ,$$

that inserted in (3.8) give us

$$\begin{array}{c} \alpha_2 + \alpha_3(m+1) + \cdots + \alpha_n(m^{n-2} + \cdots + m + 1) = \\ \frac{e_1 - d_1}{m} + \beta_2 + \beta_3(m+1) + \cdots + \beta_n(m^{n-2} + \cdots + m + 1) \; . \end{array} \tag{3.10}$$

It follows from (3.10) that $h = \frac{e_1 - d_1}{m}$ is an integer, while (3.9) implies that $|h| \leq \frac{m-1}{m}$. Therefore $h = 0$ implies $e_1 = d_1$.

Starting now from (3.10) with $d_1 = e_1$, defining $d_2 = \alpha_2 + \dots + \alpha_n$, $e_2 = \beta_2 + \dots + \beta_n$ and arguing similarly, we obtain $d_2 = e_2$. Iterating these arguments a number of times we comclude:

$$d_j = \alpha_j + \dots + \alpha_n = e_j = \beta_j + \dots + \beta_n \text{ for all } j = 1, \dots, n.$$

This implies $\alpha = \beta$ and closes our proof.

The next Theorem 3.2 shows in particular that Hörmander's condition a) in Section 2 is satisfied when it holds just for the first-order derivatives.

Theorem 3.2. *Let $P(D) = \sum_{|\alpha| \le m} c_\alpha D^\alpha$ be a partial differential operator of order $m \ge 1$ with constant coefficients $c_\alpha \in \mathbb{C}$. Then the following properties on the symbol $P(\xi) = \sum_{|\alpha| \le m} c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^n$, are equivalent (and provide hypoellipticity of $P(D)$) :*

a) for all $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \neq 0$, $\frac{P^\alpha(\xi)}{P(\xi)} \to 0$ as $|\xi| \to +\infty$.

b) $P(\xi) \neq 0$ for $|\xi| \ge C$ and for all $\theta \in \mathbb{R}^n$, $\frac{P(\xi+\theta)}{P(\xi)} \to 1$ as $|\xi| \to +\infty$.

c) $P(\xi) \neq 0$ for $|\xi| \ge C$ and for $j = 1, \dots, n$, $\frac{P^{e^j}(\xi)}{P(\xi)} \to 0$ as $|\xi| \to +\infty$.

Proof. Of course a) $\Rightarrow$ c). We just observe that since $P^\alpha(\xi)$ is a constant for some α with $|\alpha| = m$, it follows from condition $a)$ that $P(\xi) \to \infty$ for $\xi \to \infty$.
c) $\Rightarrow$ b): we consider the function $\log P(\xi + t\theta)$ of a real variable t, $0 \le t \le 1$. Since $P(\xi) \neq 0$ for large $|\xi|$, by using the principle of analytic extension we can write:

$$\log P(\xi + \theta) - \log P(\xi) = \int_0^1 \frac{d}{dt} \log P(\xi + t\theta)\, dt \ .$$

We have that

$$\left| \frac{d}{dt} \log P(\xi + t\theta) \right| \le \sum_{j=1}^n \left| \frac{\partial_{\xi_j} P(\xi + t\theta)}{P(\xi + t\theta)} \theta_j \right| \le \sum_{j=1}^n \left| \frac{\partial_{\xi_j} P(\xi + t\theta)}{P(\xi + t\theta)} \right| |\theta_j| \ ,$$

and since $|\xi + t\theta| \ge |\xi| - t|\theta| \ge |\xi| - |\theta|$, it follows from c) that

$$\frac{P(\xi + \theta)}{P(\xi)} \to 1 \text{ for } |\xi| \to +\infty \ .$$

b) $\Rightarrow$ a): from Theorem 3.1 it follows that for each $\alpha \in \mathbb{Z}_+^n$ there exists a set of real numbers $\{t_k\}_{k=1,\dots,N}$ with $N = \frac{(m+n)!}{m!n!}$ such that

$$P^\alpha(\xi) = \sum_{k=1}^N t_k P(\xi + \theta_k) \ , \text{ with } \sum_{k=1}^N t_k = 0 \text{ if } |\alpha| \neq 0 \ ,$$

where $\theta_k \in \mathbb{R}^n_+ \setminus \{0\}$ are fixed like in Theorem 3.1. Since $P(\xi) \neq 0$ for large $|\xi|$, for $|\alpha| \neq 0$ we obtain:

$$\frac{P^\alpha(\xi)}{P(\xi)} = \sum_{k=1}^{N} t_k \frac{P(\xi+\theta_k)}{P(\xi)} \to \sum_{k=1}^{N} t_k = 0 \ , \ |\xi| \to +\infty.$$

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