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ON SOME FEJÉR-TYPE INEQUALITIES FOR DOUBLE INTEGRALS

M. A. LATIF

Abstract. In this paper some new Fejér-type inequalities are established for co-ordinated convex functions.

1. Introduction

It is well known that for every convex function $f:[a,b]\to\mathbb{R}$ we have

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

These are the celebrated Hermite-Hadamard inequalities [8, 7].
The following inequalities provide the weighted generalization of (1.1) and were proved by L. Fejér (see [5]):

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx\le\frac{1}{b-a}\int_a^b f(x)p(x)dx\le\frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

where f defined as above and $p:[a,b]\to\mathbb{R}$ is non-negative integrable and symmetric about $\frac{a+b}{2}$.

The inequalities (1.1) and (1.2) have been extended, generalized and improved in a number of ways e.g. see [2, 3, 4, 6, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein.

Let us consider a bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(\alpha x+(1-\alpha)z,\alpha y+(1-\alpha)w)\le\alpha f(x,y)+(1-\alpha)f(z,w)$$

holds, for all $(x,y),(z,w)\in\Delta$ and $\alpha\in[0,1]$.

A modification for convex functions, which are also known as co-ordinated convex functions, was introduced by Dragomir in [4] (see also [2]) as follows:

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A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b], y \in [c,d]$.

A formal definition for co-ordinated convex functions was given by the M. A. Latif and M. Alomari in [9]:

Definition 1. [9, Definition 1, p.2329] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the following inequality:

$$
\begin{aligned}
& f(tx+(1-t)y, su+(1-s)w) \\
& \leq tsf(x,u)+t(1-s)f(x,w)+s(1-t)f(y,u)+(1-t)(1-s)f(y,w)
\end{aligned}
$$

holds for all $t, s \in [0,1]$ and $(x,y), (u,w) \in \Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see [4] or [2]).

In [4] an inequality of Hermite-Hadamard type for co-ordinated convex mappings on a rectangle from the plane was also established. Recently M. Alomari and M. Darus [1] (see also [16]), proved a Fejér inequality for double integrals and considered some mappings associated to it to establish some inequalities for Lipschitzian mappings.

The main purpose of the present paper is to establish some new Fejér-type inequalities for co-ordinated convex functions on rectangle from the plane.

2. Main results

We will use the following lemma to prove our results:

Lemma 1. *[1, Lemma 2.1, p.17] Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function and let*

$$
\begin{aligned}
a \leq y_1 \leq x_1 \leq x_2 \leq y_2 \leq b \quad \text{with} \quad x_1+x_2 = y_1+y_2, \\
c \leq w_1 \leq v_1 \leq v_2 \leq w_2 \leq d \quad \text{with} \quad v_1+v_2 = w_1+w_2.
\end{aligned}
$$

Then, for the convex partial mappings $f_y : [a,b] \to \mathbb{R}, f_y(t) = f(t,y)$ and $f_x : [c,d] \to \mathbb{R}, f_x(s) = f(x,s)$, for all $x \in [a,b], y \in [c,d]$, respectively, the following hold:

$$f(x_1,s)+f(x_2,s) \leq f(y_1,s)+f(y_2,s), \text{ for all } s \in [c,d],$$

and

$$f(t,v_1)+f(t,v_2) \leq f(t,w_1)+f(t,w_2), \text{ for all } t \in [a,b].$$

Throughout in this section let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and $p:[a,b]\times[c,d]\to[0,\infty)$ be integrable and symmetric about $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$. We now define the following functions on $[0,1]^2$ associated with Fejér inequality for double integrals proved in [1]:

$$\begin{aligned}I(t,s)=\frac{1}{4}\int_a^b\int_c^d\Bigg[&f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},s\frac{y+c}{2}+(1-s)\frac{c+d}{2}\right)\\&+f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2},s\frac{y+d}{2}+(1-s)\frac{c+d}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},s\frac{y+c}{2}+(1-s)\frac{c+d}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2},s\frac{y+d}{2}+(1-s)\frac{c+d}{2}\right)\Bigg]p(x,y)dydx,\end{aligned}$$

$$\begin{aligned}J(t,s)=\frac{1}{4}\int_a^b\int_c^d\Bigg[&f\left(t\frac{x+a}{2}+(1-t)\frac{3a+b}{4},s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\&+f\left(t\frac{x+a}{2}+(1-t)\frac{3a+b}{2},s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{a+3b}{2},s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{a+3b}{2},s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)\Bigg]p(x,y)dydx,\end{aligned}$$

$$\begin{aligned}M(t,s)=\frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\Bigg[&f\left(ta+(1-t)\frac{a+x}{2},sc+(1-s)\frac{y+c}{2}\right)\\&+f\left(t\frac{a+b}{2}+(1-t)\frac{x+b}{2},s\frac{c+d}{2}+(1-s)\frac{y+d}{2}\right)\Bigg]p(x,y)dydx\\&+\frac{1}{4}\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\Bigg[f\left(t\frac{a+b}{2}+(1-t)\frac{a+x}{2},s\frac{c+d}{2}+(1-s)\frac{c+y}{2}\right)\\&+f\left(tb+(1-t)\frac{x+b}{2},sd+(1-s)\frac{y+d}{2}\right)\Bigg]p(x,y)dydx\\&+\frac{1}{4}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\Bigg[f\left(ta+(1-t)\frac{a+x}{2},sd+(1-s)\frac{y+d}{2}\right)\\&+f\left(t\frac{a+b}{2}+(1-t)\frac{x+b}{2},s\frac{c+d}{2}+(1-s)\frac{y+c}{2}\right)\Bigg]p(x,y)dydx\\&\frac{1}{4}\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\Bigg[f\left(t\frac{a+b}{2}+(1-t)\frac{a+x}{2},s\frac{c+d}{2}+(1-s)\frac{y+d}{2}\right)\\&+f\left(tb+(1-t)\frac{x+b}{2},sc+(1-s)\frac{y+c}{2}\right)\Bigg]p(x,y)dydx\end{aligned}$$

and

$$N(t,s)=\frac{1}{4}\int_a^b\int_c^d\left[f\left(ta+(1-t)\frac{a+x}{2},sc+(1-s)\frac{c+y}{2}\right)\right.$$
$$+f\left(ta+(1-t)\frac{a+x}{2},sd+(1-s)\frac{y+d}{2}\right)$$
$$+f\left(tb+(1-t)\frac{x+b}{2},sc+(1-s)\frac{c+y}{2}\right)$$
$$\left.+f\left(tb+(1-t)\frac{x+b}{2},sd+(1-s)\frac{y+d}{2}\right)\right]p(x,y)dydx.$$

Theorem 1. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and the mappings $I:[0,1]^2\to\mathbb{R}$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be defined as above. Then we have*

(1) *The mapping I is co-ordinated convex on $[0,1]^2$,*
(2) *The mapping I is co-ordinated monotonic nondecreasing on $[0,1]^2$,*
(3) *We have the bounds*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)dydx=I(0,0)\le I(t,s)\le I(1,1)$$
$$=\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.$$
$$\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx. \tag{2.1}$$

Proof.

(1) It is easily observed from the co-ordinated convexity of f that I is co-ordinated convex on $[0,1]^2$.
(2) Using the simple techniques of integration, under the assumptions on p, we have that the following identity holds on $[0,1]^2$:

$$I(t,s)=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(tx+(1-t)\frac{a+b}{2},s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2},s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)\right]p(2x-a,2y-c)dydx.$$

Fix $s\in[0,1]$. Let $0\le t_1\le t_2\le 1$, then by Lemma 1, the following inequalities hold for all $x\in\left[a,\frac{a+b}{2}\right]$:

$$f\left(t_1x+(1-t_1)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)$$

$$
\begin{aligned}
&+f\left(t_1(a+b-x)+(1-t_1)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\\
&\le f\left(t_2x+(1-t_2)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_2(a+b-x)+(1-t_2)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \qquad (2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(t_1x+(1-t_1)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_1(a+b-x)+(1-t_1)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)\\
&\le f\left(t_2x+(1-t_2)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)\\
&+f\left(t_2(a+b-x)+(1-t_2)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right) \qquad (2.3)
\end{aligned}
$$

Indeed the above inequalities hold if take

$$
\begin{aligned}
y_1 &= t_2x+(1-t_2)\frac{a+b}{2},\\
x_1 &= t_1x+(1-t_1)\frac{a+b}{2},\\
x_2 &= t_1(a+b-x)+(1-t_1)\frac{a+b}{2}
\end{aligned}
$$

and

$$
y_2 = t_2(a+b-x)+(1-t_2)\frac{a+b}{2}.
$$

in Lemma 1.

Adding (2.2), (2.3), Multiplying both sides by $p(2x-a,2y-c)$ and then integrating the resulting inequality over x on $\left[a,\frac{a+b}{2}\right]$ and over y on $\left[c,\frac{c+d}{2}\right]$, we get

$$
I(t_1,s)\le I(t_2,s),
$$

for all $t_1,\ t_2\in[0,1]$ with $0\le t_1\le t_2\le 1$, for fixed $s\in[0,1]$.

We can similarly prove that

$$
I(t,s_1)\le I(t,s_2),
$$

for all $s_1,\ s_2\in[0,1]$ with $0\le s_1\le s_2\le 1$, for fixed $t\in[0,1]$.

(3) By (2) we have

$$
I(t,s)\ge I(0,s)\ge I(0,0)=f\left(\frac{a+b}{2},\frac{c+d}{2}\right)
$$

and

$$
I(t,s)\le I(1,s)\le I(1,1)=\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.
$$

$$+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\Bigg]p(x,y)dydx.$$

and thus the theorem is completely proved. □

Remark 1. If $p(x,y)=\frac{1}{(b-a)(d-c)}$, for all $(x,y)\in[a,b]\times[c,d]$, then $I(t,s)=H(t,s)$, where

$$H(t,s)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dydx,$$

for all $t,s\in[0,1]^2$. Therefore we have the inequalities (see e.g [4, Theorem 2, p. 780]:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)=H(0,0)\le H(t,s)\le H(1,1)$$
$$=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx.$$

Theorem 2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and the mappings $J:[0,1]^2\to\mathbb{R}$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be defined as above. Then we have*

(1) *The mapping J is co-ordinated convex on $[0,1]^2$,*
(2) *The mapping J is co-ordinated monotonic nondecreasing on $[0,1]^2$,*
(3) *We have the following Fejér-type inequalities:*

$$\frac{f\left(\frac{3a+b}{4},\frac{3c+d}{2}\right)+f\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)+f\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)+f\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)}{4}\int_a^b\int_c^d p(x,y)dydx$$
$$=J(0,0)\le J(t,s)\le J(1,1)=\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.$$
$$\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx. \tag{2.4}$$

Proof.

(1) The co-ordinated convexity of the mapping J follows directly from the co-ordinated convexity of f.
(2) By the simple techniques of integration, under the assumptions on p, we have the following identity:

$$J(t,s)=\frac{1}{2}\int_a^{\frac{3a+b}{4}}\int_c^d\left[f\left(tx+(1-t)\frac{3a+b}{4},s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\right.$$
$$+f\left(t\left(\frac{3a+b}{2}-x\right)+(1-t)\frac{3a+b}{4},s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+3b}{2},s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)$$
$$+f\left(t\left(\frac{3a+b}{2}-x\right)+(1-t)\frac{3a+b}{2},s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)$$

$$
\begin{aligned}
&+f\left(t\left(x+\frac{b-a}{2}\right)+(1-t)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&+f\left(t(a+b-x)+(1-t)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&+f\left(t\left(x+\frac{b-a}{2}\right)+(1-t)\frac{a+3b}{2}, s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)\\
&+f\left(tx+(1-t)\frac{3a+b}{2}, s\frac{y+d}{2}+(1-s)\frac{c+3d}{2}\right)\Bigg]\, p(2x-a,y)\,dy\,dx,
\end{aligned}
$$

for all $t,s\in[0,1]^2$.

Fix $s\in[0,1]$. Let $0\le t_1\le t_2\le 1$, then by Lemma 1, we have that the following inequalities hold for all $x\in\left[a,\frac{3a+b}{4}\right]$:

$$
\begin{aligned}
&f\left(t_1x+(1-t_1)\frac{3a+b}{4}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_1\left(\frac{3a+b}{2}-x\right)+(1-t_1)\frac{3a+b}{4}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\le f\left(t_2x+(1-t_2)\frac{3a+b}{4}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_2\left(\frac{3a+b}{2}-x\right)+(1-t_2)\frac{3a+b}{4}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&f\left(t_1x+(1-t_1)\frac{3a+b}{4}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_1\left(\frac{3a+b}{2}-x\right)+(1-t_1)\frac{3a+b}{4}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\le f\left(t_2x+(1-t_2)\frac{3a+b}{4}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_2\left(\frac{3a+b}{2}-x\right)+(1-t_2)\frac{3a+b}{4}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right),
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
&f\left(t_1\left(x+\frac{b-a}{2}\right)+(1-t_1)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_1(a+b-x)+(1-t_1)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\le f\left(t_2\left(x+\frac{b-a}{2}\right)+(1-t_2)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)\\
&\quad+f\left(t_2(a+b-x)+(1-t_2)\frac{a+3b}{2}, s\frac{y+c}{2}+(1-s)\frac{3c+d}{2}\right)
\end{aligned}
\tag{2.7}
$$

and

$$
f\left(t_1\left(x+\frac{b-a}{2}\right)+(1-t_1)\frac{a+3b}{2}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)
$$

$$+f\left(t_1(a+b-x)+(1-t_1)\frac{a+3b}{2}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)$$
$$\le f\left(t_2\left(x+\frac{b-a}{2}\right)+(1-t_2)\frac{a+3b}{2}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right)$$
$$+f\left(t_2(a+b-x)+(1-t_2)\frac{a+3b}{2}, s\frac{y+d}{2}+(1-s)\frac{3c+d}{2}\right). \tag{2.8}$$

Multiplying (2.5)-(2.8) by $p(2x-a,y)$, integrating over x on $\left[a,\frac{3a+b}{4}\right]$ and over y on $[a,b]$, we have

$$J(t_1,s)\le J(t_2,s),$$

for all $t_1,\ t_2\in[0,1]$ with $0\le t_1\le t_2\le 1$, for fixed $s\in[0,1]$.

We can similarly prove that

$$J(t,s_1)\le J(t,s_2),$$

for all $s_1,\ s_2\in[0,1]$ with $0\le s_1\le s_2\le 1$, for fixed $t\in[0,1]$.

This prove that J is co-ordinated monotonic nondecreasing on $[0,1]^2$.

(3) (2.4) follows from (2).

This completes the proof of the theorem. □

Theorem 3. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and the mappings I, $J:[0,1]^2\to\mathbb{R}$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be defined as above. Then we have*

$$I(t,s)\le J(t,s)$$

on the co-ordinates on $[0,1]^2$.

Proof. Using the simple techniques of integration, under the assumptions on p, we have that the following identities hold on $[0,1]^2$:

$$I(t,s)=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right.$$
$$+f\left(tx+(1-t)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right)$$
$$\left.+f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)\right]p(2x-a,2y-c)\,dy\,dx \tag{2.9}$$

and

$$J(t,s)=\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}, sy+(1-s)\frac{3c+d}{4}\right)\right.$$

$$+f\left(tx+(1-t)\frac{3a+b}{4}, s\left(c+d-y\right)+(1-s)\frac{c+3d}{4}\right)$$
$$+f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}, s\left(c+d-y\right)+(1-s)\frac{c+3d}{4}\right)$$
$$+f\left(tx+(1-t)\frac{3a+b}{4}, sy+(1-s)\frac{3c+d}{4}\right)\Bigg] p(2x-a,2y-c)\,dy\,dx. \tag{2.10}$$

Now for fixed $s \in [0,1]$ and by Lemma 1, we have that the following inequalities hold for all $x \in \left[a, \frac{a+b}{2}\right]$ and for all $t \in [0,1]$:

$$\begin{aligned} &f\left(tx+(1-t)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right) \\ &\quad +f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, s\left(c+d-y\right)+(1-s)\frac{c+d}{2}\right) \\ &\le f\left(tx+(1-t)\frac{3a+b}{4}, s\left(c+d-y\right)+(1-s)\frac{c+3d}{4}\right) \\ &\quad +f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}, s\left(c+d-y\right)+(1-s)\frac{c+3d}{4}\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) \\ &\le f\left(tx+(1-t)\frac{3a+b}{4}, sy+(1-s)\frac{c+3d}{4}\right) \\ &\quad +f\left(t(a+b-x)+(1-t)\frac{a+3b}{4}, sy+(1-s)\frac{c+3d}{4}\right) \end{aligned} \tag{2.12}$$

From (2.9)-(2.12) we get that

$$I(t,s) \le J(t,s),$$

for all $t \in [0,1]$ and for some fixed $s \in [0,1]$.
Similarly it can be proved that

$$I(t,s) \le J(t,s),$$

for all $s \in [0,1]$ and for some fixed $t \in [0,1]$.
This completes the proof. □

The following result contains the properties of the mapping M:

Theorem 4. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinated convex function and the mappings $M : [0,1]^2 \to \mathbb{R}$ and $p : [a,b] \times [c,d] \to [0,\infty)$ be defined as above. Then we have*

(1) *The mapping M is co-ordinated convex on $[0,1]^2$,*
(2) *The mapping M is co-ordinated monotonic nondecreasing on $[0,1]^2$,*

(3) *We have the following Fejér-type inequalities:*

$$\begin{aligned}
&\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)\right.\\
&\qquad\left.+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx\\
&\quad= M(0,0)\le M(t,s)\le M(1,1)\\
&\quad=\frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right]\int_a^b\int_c^d p(x,y)dydx \qquad (2.13)
\end{aligned}$$

Proof.

(1) Follows by the co-ordinated convexity of f.

(2) Follows by the identity

$$\begin{aligned}
M(t,s)=\int_a^{\frac{3a+b}{4}}\int_c^{\frac{3c+d}{4}}&\left[f\left(ta+(1-t)x,sc+(1-s)y\right)\right.\\
&+f\left(t\frac{a+b}{2}+(1-t)\left(x+\frac{b-a}{2}\right),s\frac{c+d}{2}+(1-s)\left(y+\frac{d-a}{2}\right)\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)\left(\frac{3a+b}{2}-x\right),s\frac{c+d}{2}+(1-s)\left(\frac{3c+d}{2}-y\right)\right)\\
&+f\left(tb+(1-t)(a+b-x),sd+(1-s)\left(c+d-y\right)\right)\\
&+f\left(ta+(1-t)x,sd+(1-s)\left(c+d-y\right)\right)\\
&+f\left(t\frac{a+b}{2}+(1-t)\left(x+\frac{b-a}{2}\right),s\frac{c+d}{2}+\left(\frac{3c+d}{2}-y\right)\right)\\
&f\left(t\frac{a+b}{2}+(1-t)\left(\frac{3a+b}{2}-x\right),s\frac{c+d}{2}+(1-s)\left(y+\frac{d-a}{2}\right)\right)\\
&\left.+f\left(tb+(1-t)(a+b-x),sc+(1-s)y\right)\right]p(2x-a,2y-c)dydx, \qquad (2.14)
\end{aligned}$$

for all $t,s\in[0,1]^2$. However the details are left to the interested reader.

(3) Using (2), we get that

$$\begin{aligned}
M(t,s)&\ge M(0,s)\ge M(0,0)\\
&=\frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f\left(\frac{a+x}{2},\frac{y+c}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx\\
&\quad+\frac{1}{4}\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d\left[f\left(\frac{a+x}{2},\frac{c+y}{2}\right)+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)dydx\\
&\quad+\frac{1}{4}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d\left[f\left(\frac{a+x}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right]p(x,y)dydx\\
&\quad\frac{1}{4}\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}\left[f\left(\frac{a+x}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right]p(x,y)dydx,
\end{aligned}$$

for all $t,s\in[0,1]^2$, which gives the first inequality in (2.13). Also we have

$$M(t,s)\le M(1,s)\le M(1,1)$$

$$
\begin{aligned}
&= \frac{1}{4}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}\left[f(a,c)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]p(x,y)\,dydx\\
&+\frac{1}{4}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f(b,d)\right]p(x,y)\,dydx\\
&+\frac{1}{4}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}\left[f(a,d)+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]p(x,y)\,dydx\\
&\frac{1}{4}\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f(b,c)\right]p(x,y)\,dydx,
\end{aligned}
$$

for all $t,s\in[0,1]^2$, which yields the second inequality in (2.13), since

$$
\begin{aligned}
\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}p(x,y)\,dydx &= \int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}p(x,y)\,dydx = \int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}p(x,y)\,dydx\\
&= \int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}p(x,y)\,dydx. \qquad \square
\end{aligned}
$$

Now we give a results which contains the properties of the mapping N.

Theorem 5. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and the mappings $N:[0,1]^2\to\mathbb{R}$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be defined as above. Then we have*

(1) *The mapping N is co-ordinated convex on $[0,1]^2$,*
(2) *The mapping N is co-ordinated monotonic nondecreasing on $[0,1]^2$,*
(3) *We have the following Fejér-type inequalities:*

$$
\begin{aligned}
&\frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right.\\
&\qquad\left.+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)\,dydx = N(0,0)\le N(t,s)\le N(1,1)\\
&\quad= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)\,dydx. \qquad (2.15)
\end{aligned}
$$

Proof. By the identity

$$
\begin{aligned}
N(t,s) = \int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}&\left[f\left(ta+(1-t)x,sc+(1-s)y\right)\right.\\
&+f\left(ta+(1-t)x,sd+(1-s)(c+d-y)\right)+f\left(tb+(1-t)(a+b-x),sc+(1-s)y\right)\\
&\left.+f\left(tb+(1-t)(a+b-x),sd+(1-s)(c+d-y)\right)\right]p(2x-a,2y-c)\,dydx,
\end{aligned}
$$

for all $t,s\in[0,1]^2$ and using a similar method to that for Theorem 1, we can show that N is co-ordinated convex, co-ordinated nondecreasing on $[0,1]^2$ and that (2.15) holds. $\square$

Remark 2. If we take $p(x,y)=\frac{1}{(b-a)(d-a)}$, for all $x,\ y\in[a,b]\times[c,d]$, in Theorem 5, then $N(t,s)=F(t,s)$, for all $t,\ s\in[0,1]^2$ (see e.g [16, Theorem 1, pg. 66]), where

$$\begin{aligned}F(t,s)=\frac{1}{4(b-a)(d-a)}\int_a^b\int_c^d&\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.\\&+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\\&+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\\&\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]dydx,\end{aligned}$$

for all $t,s\in[0,1]^2$ and (2.15) reduce to the following inequalities:

$$\begin{aligned}\frac{1}{(b-a)(d-a)}\int_a^b\int_c^d f(x,y)\,dydx=F(0,0)\le F(t,s)\le F(1,1)\\=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

Theorem 6. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a co-ordinated convex function and the mappings M, $N:[0,1]^2\to\mathbb{R}$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be defined as above. Then we have*

$$M(t,s)\le N(t,s)$$

on the co-ordinates on $[0,1]^2$.

Proof. By the identity

$$\begin{aligned}N(t,s)=\frac{1}{4}\int_a^{\frac{3a+b}{4}}\int_c^{\frac{3c+d}{4}}&[f(ta+(1-t)x,sc+(1-s)y)\\&+f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right),sc+(1-s)\left(\frac{3c+d}{2}-y\right)\right)\\&+f\left(ta+(1-t)x,sd+(1-s)\left(y+\frac{d-c}{2}\right)\right)\\&+f\left(ta+(1-t)\left(\frac{3a+b}{2}-x\right),sd+(1-s)(c+d-y)\right)\\&+f\left(tb+(1-t)(a+b-x),sc+(1-s)\left(\frac{3c+d}{2}-y\right)\right)\\&+f\left(tb+(1-t)\left(x+\frac{b-a}{2}\right),sc+(1-s)y\right)\\&+f\left(tb+(1-t)\left(\frac{3a+b}{2}-x\right),sd+(1-s)\left(\frac{3c+d}{2}-y\right)\right)\\&+f(tb+(1-t)(a+b-x),sd+(1-s)(c+d-y))]\,p(2x-a,2y-c)dydx,\end{aligned}$$

on $[0,1]^2$, (2.14) and using similar method to that for Theorem 3, we can show that $M(t,s) \leq N(t,s)$ on the co-ordinates on $[0,1]^2$. This completes the proof. □

The following is a natural consequence of Theorem 1, Theorem 2, Theorem 4 and Theorem 5:

Corollary 1. *Let f, p be defined as above. Then we have*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_a^b\int_c^d p(x,y)\,dy\,dx \\
&\leq \frac{f\left(\frac{3a+b}{4},\frac{3c+d}{2}\right)+f\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)+f\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)+f\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)}{4}\int_a^b\int_c^d p(x,y)\,dy\,dx \\
&\leq \frac{1}{4}\int_a^b\int_c^d\left[f\left(\frac{x+a}{2},\frac{y+c}{2}\right)+f\left(\frac{x+a}{2},\frac{y+d}{2}\right)+f\left(\frac{x+b}{2},\frac{y+c}{2}\right)\right. \\
&\quad\left.+f\left(\frac{x+b}{2},\frac{y+d}{2}\right)\right]p(x,y)\,dy\,dx \\
&\leq \frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right]\int_a^b\int_c^d p(x,y)\,dy\,dx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)\,dy\,dx. \qquad (2.16)
\end{aligned}$$

Remark 3. If we take $p(x,y)=\frac{1}{(b-a)(d-a)}$, for all $x,\ y\in[a,b]\times[c,d]$ in Corollary 1, then from (2.16) we have that the following inequalities hold:

$$\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{f\left(\frac{3a+b}{4},\frac{3c+d}{2}\right)+f\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)+f\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)+f\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)}{4} \\
&\leq \frac{1}{(b-a)(d-a)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\
&\leq \frac{1}{4}\left[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right] \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (2.17)
\end{aligned}$$

The inequalities in (2.17) represent a refinement of the Hermite-Hadamard type inequality for co-ordinated convex mappings proved in [4, Theorem 1, pg. 778].

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College of Science, Department of Mathematics, University of Hail, Hail-2440, Saudi Arabia.

E-mail: m_amer_latif@hotmail.com