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# STABILITY ESTIMATE FOR A STRONGLY COUPLED PARABOLIC SYSTEM

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**Abstract**. We consider an inverse source problem for a $2\times 2$ strongly coupled parabolic system. The Lipschitz stability is proved and the proof is based on the Carleman estimates with two large parameters.

## 1. Introduction

We consider the following strongly coupled parabolic system

$$\begin{cases} \partial_t u_1 = a_{11}(x,t)\Delta u_1 + a_{12}(x,t)\Delta u_2 + \widetilde{\ell}_1(u_1,\nabla u_1,u_2,\nabla u_2) + \widetilde{f}_1(x,t), \\ \partial_t u_2 = a_{21}(x,t)\Delta u_1 + a_{22}(x,t)\Delta u_2 + \widetilde{\ell}_2(u_1,\nabla u_1,u_2,\nabla u_2) + \widetilde{f}_2(x,t) \end{cases}$$

in $Q := \Omega\times(0,T)$ with initial and boundary conditions

$$\begin{cases} (u_1,u_2)\,|_{\partial\Omega\times(0,T)} = 0, \\ (u_1,u_2)\,|_{t=0} = (u_{01}(x),u_{02}(x)) \text{ in } \Omega\subseteq\mathbb{R}^n. \end{cases}$$

Here $\Omega\subseteq\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. $A(x,t) := (a_{ij})_{2\times 2}$ is a smooth positive matrix with different eigenvalues. $\widetilde{\ell}_i(i=1,2)$ are smooth linear functions and $\widetilde{f}_i(i=1,2)$ are smooth unknown functions. It can be proved (see Appendix) that there exists an invertible matrix $P(x,t)=(p_{ij})_{2\times 2}$ such that

$$P^{-1}AP = \begin{pmatrix} \mu_1 & 0 \\ \mu_0 & \mu_2 \end{pmatrix}$$

with $\mu_1>0,\ \mu_2>0$. Let

$$\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = P^{-1}\begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{pmatrix},$$

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and similarly, we denote $\ell_i(u,\nabla u,v,\nabla v)$ $(i=1,2)$ the linear function of $u,\nabla u,v,\nabla v$ with the coefficients depending on $(x,t)$. Therefore, it is natural to consider the following system

$$\begin{cases} \partial_t u = \mu_1 \Delta u + \ell_1(u,\nabla u,v,\nabla v) + f_1(x,t), \\ \partial_t v = \mu_0 \Delta u + \mu_2 \Delta v + \ell_2(u,\nabla u,v,\nabla v) + f_2(x,t), \\ u = v = 0 \ \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Our inverse problem is to find $\{f_1(x,t), f_2(x,t)\}$ from the input data

$$(u,v)\mid_{\omega\times(0,T)} \text{ and } (u,v)(x,t_0),\ x\in\Omega,\ t_0\in(0,T). \tag{1.2}$$

Here $\omega \subset\subset \Omega$ is a given subdomain. Our aim is to prove the Lipschitz stability of the inverse source problem. The key ingredient is a Carleman estimate with two large parameters due to Yamamoto [1]. The pioneering paper for the inverse problem by Carleman estimate was obtained by Bukhgeim and Klibanov [2]. For the related work, Klibanov [3] got estimates of initial conditions of parabolic equations and inequalities via lateral cauchy data. The first Lipschitz stability result of an inverse source problem for a parabolic equation was obtained by Imanuvilov and Yamamoto [4]. Moreover the method of [4] was generalized to the inverse source problem of Navier–Stokes equation [5] and the Boussinesq system [6], respectively. Very recently, Benabdallah–Cristofol–Gaitan–Yamamoto [7] study our inverse problem (1.1) with $\mu_0 = 0$. We note that the method of [7] can not be adapted to our system.

We make the following assumptions.

$$\begin{cases} (H1)\ |\partial_t f_i(x,t)| \le c\,|f_i(x,t_0)|\ \ (i=1,2) \quad (x,t)\in\overline{\Omega}\times[0,T]. \\ (H2)\ \mu_0,\ \mu_1,\ \mu_2 \in C^2(\overline{\Omega}\times[0,T]) \quad \mu_1>0,\ \mu_2>0. \end{cases}$$

Each $\ell_i(u,\nabla u,v,\nabla v)$ denotes a first order different differential operator acting on $u$ and on $v$ with $C^2$ coefficients depending on $(x,t)$.

**Remark 1.1.** $(H1)$ is also assumed in [4]. Therefore, there exists a positive constant $c$ such that

$$|f_i(x,t)| + |\partial_t f_i(x,t)| \le c\,|f_i(x,t_0)|\ \ (i=1,2) \tag{1.3}$$

for $(x,t)\in\overline{\Omega}\times[0,T]$. In [4], $f_i(x,t) = p_i(x)R_i(x,t)$ where $R_i(x,t)$ is given, and $R_i(x,t_0)\neq 0$. Then we are in a position to state our main result in this paper.

**Theorem 1.1.** *Let $(H1)$ and $(H2)$ be satisfied. Then there exists a positive constant $c := c(\Omega,T,t_0)$ such that*

$$\|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)} \le c\{\|u(\cdot,T/2)\|_{H^2} + \|v(\cdot,T/2)\|_{H^2} + \|u\|_{H^1(0,T;L^2(\omega))} + \|v\|_{H^1(0,T;L^2(\omega))}\}.$$

**Remark 1.2.** If $t_0 = 0$, then the corresponding inverse problem for a parabolic equation is open (cf Isakov [8, 9]); our inverse problem with $t_0 = 0$ is also open.

**Remark 1.3.** We can not prove a similar result for an $m \times n$ strongly coupled parabolic system for $m, n \geq 3$. It is an open problem posed in [1] to prove a Carleman estimate for $n \times n$ strongly coupled parabolic system for $n \geq 2$. Thus we solve the open problem when $n = 2$.

## 2. Key Carleman estimate

For our Carleman estimate we need a weight function with special properties. The existence of such a function is proved in [10, 11].

**Lemma 2.1.** *Let $\omega_0$ be an arbitrary fixed sub-domain of $\Omega$ such that $\overline{\omega_0} \subset \omega$. Then there exists a function $\eta \in C^2(\overline{\Omega})$ such that*

$$\eta(x) > 0 \ \ in\, \Omega,\ \eta|_{\partial\Omega} = 0,\ \ |\nabla\eta(x)| > 0 \ \ in\, \overline{\Omega\setminus\omega_0}. \tag{2.1}$$

Let $\Omega := \{x;\ \ |x| < 1\}$ and $\omega_0 := \{x;\ \ |x| < \frac{1}{2}\}$. Then $\eta(x) = 1 - |x|^2$ satisfies (2.1). Set

$$\varphi(x,t) = \frac{e^{\lambda\eta}}{t(T-t)},\ \ \alpha(x,t) = \frac{e^{\lambda\eta(x)} - e^{2\lambda\|\eta\|_{C(\overline{\Omega})}}}{t(T-t)},$$

where $\lambda > 0$, then we have the following Carleman estimates with two large parameters which were proved in [12].

**Lemma 2.2.** *Let*

$$\begin{cases} \partial_t y - a(x,t)\Delta y + b(x,t)\cdot\nabla y + c(x,t)\,y = g \ \ in\, Q := \Omega\times(0,T), \\ y = 0 \ \ on\, \partial\Omega\times(0,T), \end{cases}$$

*where $\|a\|_{C^1(\overline{Q})} + \|b\|_{L^\infty(Q)} + \|c\|_{L^\infty(Q)} \leq M$, $g \in L^2(Q)$. Then there exists a number $\lambda_0 > 0$ such that for an arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) \geq 0$ satisfying: there exist a constant $c := c(s_0, \lambda_0) > 0$ such that*

$$\begin{aligned} &\int_Q \{(s\varphi)^{p-1}(|\partial_t y|^2 + \Sigma_{i,j=1}^n |\partial_i\partial_j y|^2) \\ &\quad + (s\varphi)^{p+1}\lambda^2|\nabla y|^2 + (s\varphi)^{p+3}\lambda^4 y^2\} e^{2s\alpha}\,dxdt \\ &\leq c\int_Q (s\varphi)^p |g|^2 e^{2s\alpha}\,dxdt + c\int_{\omega\times(0,T)} (s\varphi)^{p+3}\lambda^4 y^2 e^{2s\alpha}\,dxdt, \end{aligned} \tag{2.2}$$

*for all $s > s_0$, $p = 0, 1, 2$.*

## 3. Proof of Theorem 1.1

First we assume that $t_0 = T/2$ without loss of generality by changing the scale of t. Applying Lemma 2.2 $(p = 2)$ to the solution $u$ of (1.1), we have the estimate

$$\int_Q \{s\varphi(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2) + (s\varphi)^3\lambda^2|\nabla u|^2 + (s\varphi)^5\lambda^4 u^2\}e^{2s\alpha}dxdt$$
$$\leq c\int_Q (s\varphi)^2(f_1^2 + |\nabla v|^2 + |v|^2)e^{2s\alpha}dxdt + c\int_{\omega\times(0,T)} (s\varphi)^5\lambda^4 u^2 e^{2s\alpha}dxdt. \tag{3.1}$$

Similarly, applying Lemma 2.2 $(p = 1)$ to the solution $v$ of (1.1), we get

$$\int_Q \{|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 + (s\varphi)^2\lambda^2|\nabla v|^2 + (s\varphi)^4\lambda^4 v^2\}e^{2s\alpha}dxdt$$
$$\leq c\int_Q s\varphi(f_2^2 + |\Delta u|^2 + |\nabla u|^2 + |u|^2)e^{2s\alpha}dxdt$$
$$+ c\int_{\omega\times(0,T)} (s\varphi)^4\lambda^4 v^2 e^{2s\alpha}dxdt. \tag{3.2}$$

By $\varphi \geq \frac{1}{t(T-t)} \geq \frac{4}{T^2}$, $(2c)\times(3.1) + (3.2)$ and $\lambda$ sufficiently large, we deduce that

$$\int_Q \{s\varphi(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u|^2) + (s\varphi)^3\lambda^2|\nabla u|^2 + (s\varphi)^5\lambda^4 u^2$$
$$+ (|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2) + (s\varphi)^2\lambda^2|\nabla v|^2 + (s\varphi)^4\lambda^4 v^2\}e^{2s\alpha}dxdt$$
$$\leq c\int_Q [(s\varphi)^2 f_1^2 + s\varphi f_2^2]e^{2s\alpha}dxdt + \tilde{c}\int_{\omega\times(0,T)} (u^2 + v^2)dxdt. \tag{3.3}$$

Taking $\partial_t$ to the first equation of (1.1), we see that

$$\partial_t u_t = \mu_1\Delta u_t + \partial_t\mu_1\Delta u + \partial_t\ell_1 + \partial_t f_1. \tag{3.4}$$

Applying Lemma 2.2 $(p = 1)$ to (3.4), we have

$$\int_Q \{|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u_t|^2 + (s\varphi)^2\lambda^2|\nabla u_t|^2 + (s\varphi)^4\lambda^4 u_t^2\}e^{2s\alpha}dxdt$$
$$\leq c\int_Q s\varphi(|\partial_t f_1|^2 + |\Delta u|^2 + |\nabla u|^2 + |u|^2 + |v|^2 + |\nabla v|^2$$
$$+ |v_t|^2 + |\nabla v_t|^2)e^{2s\alpha}dxdt + \tilde{c}\int_{\omega\times(0,T)} |u_t|^2 dxdt. \tag{3.5}$$

Taking $\partial_t$ to the second equation of (1.1), we find that

$$\partial_t v_t = \mu_0\Delta u_t + \mu_2\Delta v_t + \partial_t\mu_0\Delta u + \partial_t\mu_2\Delta v + \partial_t\ell_2 + \partial_t f_2. \tag{3.6}$$

Applying Lemma 2.2 ($p=0$) to (3.6), we have

$$\int_Q \{\frac{1}{s\varphi}(|\partial_t^2 v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v_t|^2) + s\varphi\lambda^2 |\nabla v_t|^2 + (s\varphi)^3\lambda^4 v_t^2)\} e^{2s\alpha} dxdt$$
$$\le c\int_Q (|\partial_t f_2|^2 + |\Delta u_t|^2 + |\nabla u_t|^2 + |u_t|^2 + |\Delta u|^2 + |\nabla u|^2 + |u|^2$$
$$+ |\Delta v|^2 + |\nabla v|^2 + |v|^2) e^{2s\alpha} dxdt + \widetilde{c}\int_{\omega\times(0,T)} |v_t|^2 dxdt. \tag{3.7}$$

Using $(2c)\times(3.5)+(3.7)$, $\phi \ge \frac{4}{T^2}$ and $\lambda$ sufficiently large, we find that

$$\int_Q \{|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u_t|^2 + (s\varphi)^2\lambda^2 |\nabla u_t|^2 + (s\varphi)^4\lambda^4 u_t^2$$
$$+ \frac{1}{s\varphi}(|\partial_t^2 v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v_t|^2) + s\varphi\lambda^2 |\nabla v_t|^2 + (s\varphi)^3\lambda^4 v_t^2\} e^{2s\alpha} dxdt$$
$$\le c\int_Q [s\varphi |\partial_t f_1|^2 + |\partial_t f_2|^2 + s\varphi(|\Delta u|^2 + |\nabla u|^2 + |u|^2 + |v|^2$$
$$+ |\nabla v|^2) + |\Delta v|^2] e^{2s\alpha} dxdt + \widetilde{c}\int_{\omega\times(0,T)} (|u_t|^2 + |v_t|^2) dxdt. \tag{3.8}$$

Combining (3.3), (3.8), $\phi \ge \frac{4}{T^2}$ and $s$ sufficiently large, we obtain

$$I := \int_Q \{|\partial_t^2 u|^2 + \sum_{i,j=1}^n |\partial_i\partial_j u_t|^2 + (s\varphi)^2\lambda^2 |\nabla u_t|^2 + (s\varphi)^4\lambda^4 u_t^2$$
$$+ s\varphi \sum_{i,j=1}^n |\partial_i\partial_j u|^2 + (s\varphi)^3\lambda^2 |\nabla u|^2 + (s\varphi)^5\lambda^4 u^2$$
$$+ \frac{1}{s\varphi}(|\partial_t^2 v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v_t|^2) + s\varphi\lambda^2 |\nabla v_t|^2 + (s\varphi)^3\lambda^4 v_t^2$$
$$+ \sum_{i,j=1}^n |\partial_i\partial_j v|^2 + (s\varphi)^2\lambda^2 |\nabla v|^2 + (s\varphi)^4\lambda^4 v^2\} e^{2s\alpha} dxdt$$
$$\le c\int_Q [(s\varphi)^2 f_1^2 + s\varphi f_2^2 + s\varphi(\partial_t f_1)^2 + (\partial_t f_2)^2] e^{2s\alpha} dxdt$$
$$+ \widetilde{c}\int_{\omega\times(0,T)} (u^2 + u_t^2 + v^2 + v_t^2) dxdt$$
$$\le c\int_Q [(s\varphi)^2 f_1^2(x,T/2) + s\varphi f_2^2(x,T/2)] e^{2s\alpha} dxdt$$
$$+ \widetilde{c}\int_{\omega\times(0,T)} (u^2 + u_t^2 + v^2 + v_t^2) dxdt. \tag{3.9}$$

On the other hand, in terms of $u_t^2(x,0)e^{2s\alpha(x,0)} = 0$, $v_t^2(x,0)e^{2s\alpha(x,0)} = 0$, $|\alpha_t| \le c\varphi^2$, and equations (1.1) (1.2) at time $t = T/2$, we obtain

$$\int_\Omega [s^2 f_1^2(x,T/2) + s f_2^2(x,T/2)] e^{2s\alpha(x,T/2)} dx$$

$$
\begin{aligned}
&\le c\int_\Omega (s^2 \mid \partial_t u(x,T/2)\mid^2 + s\mid \partial_t v(x,T/2)\mid^2) e^{2s\alpha(x,T/2)}dx \\
&\quad + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&= c\int_0^{\frac{T}{2}} \frac{\partial}{\partial_t}\int_\Omega [s^2 u_t^2(x,t) + s v_t^2(x,t)] e^{2s\alpha(x,t)}dx \\
&\quad + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\le \int_0^{\frac{T}{2}}\int_\Omega (2s^3\partial_t\alpha\, u_t^2 + 2s^2 u_t u_{tt} + 2s^2\partial_t\alpha\, v_t^2 + 2s v_t v_{tt}) e^{2s\alpha}dxdt \\
&\quad + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\le \int_Q (2s^3\varphi^2 u_t^2 + 2s^2 u_t u_{tt} + 2s^2\varphi^2 v_t^2 + 2s v_t v_{tt}) e^{2s\alpha}dxdt \\
&\quad + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\le cI + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\le c\int_Q [(s\varphi)^2 f_1^2(x,T/2) + s\varphi f_2^2(x,T/2)] e^{2s\alpha}dxdt \\
&\quad + c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\quad + \widetilde{c}\int_{\omega\times(0,T)} (u^2 + u_t^2 + v^2 + v_t^2)dxdt. \qquad (3.10)
\end{aligned}
$$

By the same calculation as that in [4], we deduce that

$$
\begin{aligned}
&\int_Q [(s\varphi)^2 f_1^2(x,T/2) + s\varphi f_2^2(x,T/2)] e^{2s\alpha}dxdt \\
&\quad \le \frac{c}{\sqrt{s}}\int_\Omega [s^2 f_1(x,T/2) + s f_2^2(x,T/2)] e^{2s\alpha(x,T/2)}dx. \qquad (3.11)
\end{aligned}
$$

Combining (3.10), (3.11) and $s$ sufficiently large, we arrive at

$$
\begin{aligned}
\int_Q (f_1^2 + f_2^2)dxdt &\le c\int_\Omega (s^2 f_1^2(x,T/2) + s f_2^2(x,T/2)) e^{2s\alpha(x,T/2)}dx \\
&\le c\parallel u(\cdot,T/2)\parallel_{H^2}^2 + c\parallel v(\cdot,T/2)\parallel_{H^2}^2 \\
&\quad + \widetilde{c}\int_{\omega\times(0,T)} (u^2 + u_t^2 + v^2 + v_t^2)dxdt.
\end{aligned}
$$

Hence we complete the proof. □

## 4. Appendix

In this appendix, we prove that there exists a smooth and invertible matrix $P = P(x,t)$ such that

$$
P^{-1}AP = P^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} P = \begin{pmatrix} \mu_1 & 0 \\ \mu_0 & \mu_2 \end{pmatrix},
$$

where $\mu_1$ and $\mu_2$ are eigenvalues of $A$. If $a_{11}(x,t) \geq a_{22}(x,t)$, we define

$$\mu_1 = \frac{a_{11}+a_{22}+\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2},$$
$$\mu_2 = \frac{a_{11}+a_{22}-\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2}.$$

If $a_{11}(x,t) < a_{22}(x,t)$, we define

$$\mu_1 = \frac{a_{11}+a_{22}-\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2},$$
$$\mu_2 = \frac{a_{11}+a_{22}+\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2}.$$

Let

$$\mu_0(x,t) = a_{21}(x,t) \quad \text{and} \quad P = \begin{pmatrix} 1 & \frac{a_{12}}{\mu_2-a_{11}} \\ 0 & 1 \end{pmatrix},$$

we can easily check that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & \frac{a_{12}}{\mu_2-a_{11}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{a_{12}}{\mu_2-a_{11}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mu_1 & 0 \\ a_{21} & \mu_2 \end{pmatrix}.$$

and

$$\mu_2(x,t) - a_{11}(x,t) \neq 0.$$

The proof is complete. □

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