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COMMON FIXED POINTS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. A common fixed point theorem for noncommuting generalized asymptotically nonexpansive mappings has been obtained in convex metric spaces. As an application, a result on the set of best approximation is also derived for such class of mappings. The proved results unify and extend some of the known results on the subject.

1. Introduction

Common fixed points of two commuting mappings satisfying some contractive or non-expansive type conditions have been studied by many researchers (see [1]-[4],[7],[9]-[12] and references cited therein). The introduction of noncommuting mappings such as weakly commuting, R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting mappings, was a turning point in the fixed point arena. A wider class of nonexpansive mapings, known as asymptotically nonexpansive mappings, was introduced by Goebel and Kirk [5]. Vijayaraju and Hemavathy [12] proved some common fixed point theorems and approximation results by extending the results of Beg et al. [2] to generalized asymptotically S-nonexpansive and C_q-commuting mappings in normed linear spaces. This paper deals with the study of common fixed point theorem for generalized asymptotically S-nonexpansive and noncommuting mappings in convex metric spaces. As an application, we also establish a result on the set of best approximation. The results proved in the paper unify and extend some known results in the literature.

2. Definitions and Preliminaries

For a metric space (X,d), a continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1-\lambda) d(u, y)$$

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holds for all $u \in X$. The metric space (X,d) together with a convex structure is called a *convex metric space* [13].

A subset K of a convex metric space (X,d) is said to be a *convex set* [13] if $W(x,y,\lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

A set K is said to be p*-starshaped* (see [6]) where $p \in K$, provided $W(x,p,\lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$ i.e. the segment

$$[p,x] = \{W(x,p,\lambda) : 0 \le \lambda \le 1\}$$

joining p to x is contained in K for all $x \in K$. K is said to be *starshaped* if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A convex metric space (X,d) is said to satisfy *Property (I)* [6] if for all $x, y, q \in X$ and $\lambda \in [0,1]$,

$$d(W(x,q,\lambda), W(y,q,\lambda)) \le \lambda d(x,y).$$

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [6], [13]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset M of a metric space (X,d) and $x \in X$, an element $y \in M$ is said to be a *best approximant* of x in M or a *best M-approximant* to x if $d(x,y) = d(x,M) \equiv \inf\{d(x,y) : y \in M\}$. The set of all such $y \in M$ is denoted by $P_M(x)$.

For a convex subset M of a convex metric space (X,d), a mapping $g : M \to X$ is said to be *affine* if for all $x, y \in M$, $g(W(x,y,\lambda)) = W(gx, gy, \lambda)$ for all $\lambda \in [0,1]$. g is said to be *affine with respect to $p \in M$* if $g(W(x,p,\lambda)) = W(gx, gp, \lambda)$ for all $x \in M$ and $\lambda \in [0,1]$.

Suppose M is a nonempty subset of a metric space (X,d) and S, T are self mappings of M. A point $x \in M$ is a *common fixed (coincidence) point* of S and T if $x = Sx = Tx (Sx = Tx)$. The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S,T)$ (respectively, $C(S,T)$). The mappings $T, S : M \to M$ are said to be

(i) *commuting* on M if $STx = TSx$ for all $x \in M$;

(ii) R*-weakly commuting* on M if there exists a real number $R > 0$ such that $d(TSx, STx) \le R\, d(Tx, Sx)$ for all $x \in M$.

(iii) *weakly compatible* if they commute at their coincidence points,i.e., if $STx = TSx$ whenever $Sx = Tx$;

(iv) *asymptotically S-nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers in $[1,\infty)$ with $k_n \ge k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \le k_n d(Sx, Sy)$, for all $x, y \in M$.

If S =identity mapping in (iv), then T is said to be an *asymptotically nonexpansive mapping* and further, if $k_n = 1$ for all $n \in \mathbb{N}$, then T is known as *nonexpansive*.

T is said to be *uniformly asymptotically regular* on M if, for each $\epsilon > 0$, there exists a positive integer N such that $d(T^n(x), T^n(y)) < \epsilon$ for all $n \geq N$ and for all $x, y \in M$.

Suppose (X, d) is a convex metric space, M a q-starshaped subset with $q \in F(S) \cap M$ and is both T- and S-invariant. Then T and S are called

(i) *R-subweakly commuting* (see [11]) on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R\, \text{dist}(Sx, W(Tx, q, k))$, $k \in [0, 1]$;

(ii) *C_q-commuting* [1] if $STx = TSx$ for all $x \in C_q(S, T)$, where $C_q(S, T) = \cup\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k x = \{W(Tx, q, k) : 0 \leq k \leq 1\}$.

(iii) *generalized asymptotically S-nonexpansive* (see [12]) if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that

$$d(T^n x, T^n y) \leq k_n \max\{d(Sx, Sy), \text{dist}(Sx, [T^n x, q]), \text{dist}(Sy, [T^n y, q]), \frac{1}{2}[\text{dist}(Sx, [T^n y, q]) + \text{dist}(Sy, [T^n x, q])]\}$$

for all $x, y \in M$.

Clearly, C_q-commuting maps are weakly compatible. However, converse is not true.

Example 2.1 ([1])**.** Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = x^2$ for all $x \neq 2$ and $T2 = 1$; and $Sx = 2x$ for all $x \in M$. Then M is q-starshaped with $q = 0$, $C(T, S) = \{0\}$ and $C_q(T, S) = \{0\} \cup [2, \infty)$. Moreover, T and S are weakly compatible but not C_q-commuting.

Clearly, R-subweakly commuting mappings are C_q-commuting but converse does not hold.

Example 2.2 ([1])**.** Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$; and $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x \geq 1$. Then M is q-starshaped with $q = 1$, and $C_q(T, S) = [1, \infty)$. Moreover S and T are C_q-commuting but not R-subweakly commuting.

If T and S are C_q-commuting on M, then $ST^n x = T^n Sx$ for all $x \in C_q(S, T^n)$, where $C_q(S, T^n) = \cup\{C(S, T_{k_n}) : 0 \leq k_n \leq 1\}$ and $T_{k_n} x = \{W(Tx, q, k_n) : 0 \leq k_n \leq 1\}$.

Example 2.3 ([12])**.** Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [1, \infty)$. Then M is q-starshaped with $q = 1$. Define $T, S : M \to M$ by $Tx = 2x^2 - 1$ and $Sx = 4x - 3$ for all $x \in M$. Then $C_q(T, S) = [1, \infty)$ and the pair is C_q-commuting on M. Now $C_q(T, S^n) = [1, \infty)$ for all $n \geq 1$. It is easy to verify that $TS^n x = S^n Tx$ for all $x \in c_q(T, S^n)$ and for each n.

3. Main Results

To prove the main results, we need the following lemma. For normed linear spaces this lemma was proved in [12] and the proof can be easily extended to metric spaces.

Lemma 3.1. *Let M be a nonempty closed subset of a metric space (X,d). Let $f,T: M \to M$ be self mappings, $q \in F(f)$ and $T(M\backslash\{q\}) \subset f(M)\backslash\{q\}$. Suppose there exists $k \in (0,1)$ such that*

$$d(Tx,Ty) \le k \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\right\} \qquad (3.1)$$

for all $x,y \in M$. Further, if T is continuous, $cl\ [T(M\backslash\{q\})]$ is complete, f and T are weakly compatible on $M\backslash\{q\}$, then $F(f)\cap F(T)$ is singleton.

The following theorem extends and generalizes the corresponding results of Al-Thagafi and Shahzad ([1]-Theorem 2.2), Hussain and Rhoades ([7]-Theorem 2.2) and of Vijayaraju and Hemavathy ([12]-Theorem 3.1).

Theorem 3.2. *Let M be a nonempty subset of a convex metric space (X,d) with Property (I) and T and f are self mappings of M. Suppose that M is q-starshaped with $q \in F(f)$ and f is continuous and affine with respect to q. If T and f are C_q-commuting on $M\backslash\{q\}$, $cl\ [T(M\backslash\{q\})]$ is compact, $cl\ [T(M)] \subseteq f(M)\backslash\{q\}$, and T is continuous, uniformly asymptotically regular and generalized asymptotically f-nonexpansive i.e. it satisfies,*

$$\begin{aligned} d(T^n x, T^n y) \le \mu_n \max\{&d(fx,fy), \mathrm{dist}(fx,[T^n x,q]), \mathrm{dist}(fy,[T^n y,q]), \\ &\frac{1}{2}[\mathrm{dist}(fx,[T^n y,q]) + \mathrm{dist}(fy,[T^n x,q])]\} \end{aligned} \qquad (3.2)$$

for all $x,y \in M$, where μ_n is a sequence of real numbers with $\mu_n \ge 1$, and $\lim \mu_n = 1$, then $F(T)\cap F(f) \ne \emptyset$.

Proof. Let (λ_n) be a sequence of real numbers such that $0 \le \lambda_n < 1$ and $\lambda_n \to 1$. Take $k_n = \frac{\lambda_n}{\mu_n}$, then (k_n) is a sequence of real numbers such that $0 \le k_n < 1$, and $k_n \to 1$. Define T_n as $T_n x = W[T^n x, q, k_n]$ for all $x \in M$ and for each $n \ge 1$. As M is q-starshaped, f is affine with respect to q and $cl[T(M\backslash\{q\})] \subseteq f(M)\backslash\{q\}$, T_n is a self mapping of M and $cl[T_n(M\backslash\{q\})] \subseteq f(M)\backslash\{q\}$ for each n. Since T and f are C_q-commuting, $fT^n x = T^n f x$ for all $x \in C_q(f,T^n)$. As f is affine with respect to q, it follows that for each $x \in C_q(f,T)$, $fT_n x = f(W[T^n x,q,k_n]) = W[fT^n x, fq, k_n] = W[T^n fx, fq, k_n] = T_n fx$. Thus $fT_n x = T_n fx$ for each $x \in C(f,T_n) \subseteq C_q(f,T^n)$. Hence the pair f and T_n are weakly compatible for all n. Further, we have

$$d(T_n x, T_n y) = d(W[T^n x, q, k_n], W[T^n y, q, k_n])$$

$$\begin{aligned}
&\le k_n d(T^n x, T^n y)\\
&\le k_n \mu_n \max\{d(fx, fy), \mathrm{dist}(fx, [T^n x, q]), \mathrm{dist}(fy, [T^n y, q]),\\
&\quad \frac{1}{2}[\mathrm{dist}(fx, [T^n y, q]) + \mathrm{dist}(fy, [T^n x, q])]\}\\
&\le \lambda_n \max\{d(fx, fy), d(fx, T_n x), d(fy, T_n y), \frac{1}{2}[d(fx, T_n y) + d(fy, T_n x)]\}
\end{aligned}$$

for all $x, y \in M$.

As $cl[T(M \backslash \{q\})]$ is compact, each $cl[T_n(M \backslash \{q\})]$ is also compact. By Lemma 3.1, there exists $x_n \in M$ such that $fx_n = T_n x_n = x_n$. Since $\{T^n x_n\}$ is a sequence in the compact set $cl\ [T(M \backslash \{q\})]$, there exists a subsequence $\{T^{n_i} x_{n_i}\}$ of $\{T^n x_n\}$ such that $\{T^{n_i} x_{n_i}\} \to z$, for some $z \in cl\ [T(M \backslash \{q\})]$. Moreover,

$$x_{n_i} = fx_{n_i} = T_{n_i} x_{n_i} = W[T^{n_i} x_{n_i}, q, k_{n_i}] \to z.$$

Since f is continuous, $x_{n_i} = fx_{n_i} \to fz$. By the uniqueness of the limit $z = fz$.

As T is continuous, $T^{n_i} x_{n_i} \to T^{n_i} z$. Again by the uniqueness of the limit, we have $\lim T^{n_i} z = z$ and $\lim T^{n_i+1} z = Tz$. Hence it follows that

$$\begin{aligned}
d(z, Tz) &\le d(z, T^{n_i} z) + d(T^{n_i} z, T^{n_i+1} z) + d(T^{n_i+1} z, Tz)\\
&\to 0.
\end{aligned}$$

Therefore $Tz = z = fz$. Hence $F(f) \cap F(T) \neq \emptyset$. □

Remark 1. When T and f are R-subweakly commuting and T is f-nonexpansive, results similar to Theorem 3.2 were proved by Shahzad [11] for normed linear spaces and by the authors in [9] for convex metric spaces.

The following corollary extends Theorem 2.2 of Al-Thagafi and Shahzad [1] to asymptotically nonexpansive mappings. For normed linear spaces, the following result is given in [12].

Corollary 3.3. *Let T and f be self mappings of a nonempty subset M of a convex metric space (X, d) with Property (I). Suppose that M is q-starshaped with $q \in F(f)$, f is continuous and affine with respect to q, $cl\ [T(M \backslash \{q\})]$ is compact and $cl\ [T(M)] \subseteq f(M) \backslash \{q\}$. If T and f are C_q-commuting on $M \backslash \{q\}$ and T is uniformly asymptotically regular and asymptotically nonexpansive with sequence $\{\mu_n\}$, where $\{\mu_n\}$ is a sequence of real numbers with $\mu_n \ge 1$ and $\mu_n \to 1$, then $F(T) \cap F(f) \neq \emptyset$.*

We shall now give an application of Theorem 3.2 to the set of best approximation. For this, we need the following result.

Proposition 3.4. *If M is a subset of a convex metric space (X,d), $u \in X \setminus M$ and $y \in P_M(u)$, then the line segment $\{W(y,u,\lambda) : 0 < \lambda < 1\}$ and the set M are disjoint.*

Proof. Since $y \in P_M(u)$, consider

$$\begin{aligned} d(u, W(y,u,\lambda)) &\leq \lambda\, d(u,y) \\ &< d(u,M), \text{ for every } 0 < \lambda < 1. \end{aligned}$$

This implies that $W(y,u,\lambda) \notin M$ for any λ, $0 < \lambda < 1$. Therefore the line segment $\{W(y,u,\lambda) : 0 < \lambda < 1\}$ and the set M are disjoint. □

Taking $G = M \setminus \{q\}$ for some $q \in M$, we have the following result.

Theorem 3.5. *Let M be a nonempty subset of a convex metric space (X,d) with Property (I) and T and f are self mappings of X such that $T(\partial G \cap G) \subseteq G$ and $u \in F(T) \cap F(f)$ for some $u \in X \setminus M$, where ∂G denotes boundary of G. Suppose that $P_G(u)$ is closed and q-starshaped and f is affine with respect to $q \in F(f)$ with $f(P_G(u)) = P_G(u)$. If T and f are continuous, C_q-commuting on $P_G(u) \cup \{u\}$ satisfying $d(Tx,Tu) \leq d(fx,fu)$, $cl\ T(P_G(u)$ is compact, and T is uniformly asymptotically regular and generalized asymptotically f-nonexpansive mapping for all $x, y \in P_G(u)$, then $P_G(u) \cap F(T) \cap F(f) \neq \emptyset$.*

Proof. Let $x \in D = P_G(u)$, then for any $k \in (0,1]$, we have

$$d(W(u,x,k),u) \leq k d(u,u) + (1-k) d(x,u) = (1-k) d(x,u) < \text{dist}(u,G).$$

It follows from Proposition 3.4 that the open line segment $\{W(u,x,\lambda) : 0 < \lambda < 1\}$ and the set G are disjoint. Thus x is not in the interior of G and so $x \in \partial G \cap G$. Since $T(\partial G \cap G) \subset G$, Tx must be in G. Also, $f(P_G(u)) = P_G(u)$, $fx \in P_G(u)$, $u \in F(T) \cap F(f)$, we have

$$d(Tx,u) = d(Tx,Tu) \leq d(fx,fu) = d(fx,u) \leq \text{dist}(u,G).$$

This implies that $Tx \in P_G(u)$. Consequently, $D = P_G(u)$ is T invariant. Hence by Theorem 3.2, there exists $z \in P_G(u)$ such that $P_G(u) \cap F(T) \cap F(f) \neq \emptyset$. □

Remark 2. Theorem 3.5 extends the corresponding results of [8], [9], [10] and [11] to generalized asymptotically f-nonexpansive mappings.

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