COMMON FIXED POINTS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. A common fixed point theorem for noncommuting generalized asymptotically nonexpansive mappings has been obtained in convex metric spaces. As an application, a result on the set of best approximation is also derived for such class of mappings. The proved results unify and extend some of the known results on the subject.

1. Introduction

Common fixed points of two commuting mappings satisfying some contractive or non-expansive type conditions have been studied by many researchers (see [1]-[4],[7],[9]-[12] and references cited therein). The introduction of noncommuting mappings such as weakly commuting, $R$-weakly commuting, $R$-subweakly commuting, compatible, weakly compatible and $C_q$-commuting mappings, was a turning point in the fixed point arena. A wider class of nonexpansive mappings, known as asymptotically nonexpansive mappings, was introduced by Goebel and Kirk [5]. Vijayaraju and Hemavathy [12] proved some common fixed point theorems and approximation results by extending the results of Beg et al. [2] to generalized asymptotically $S$-nonexpansive and $C_q$-commuting mappings in normed linear spaces. This paper deals with the study of common fixed point theorem for generalized asymptotically $S$-nonexpansive and noncommuting mappings in convex metric spaces. As an application, we also establish a result on the set of best approximation. The results proved in the paper unify and extend some known results in the literature.

2. Definitions and Preliminaries

For a metric space $(X, d)$, a continuous mapping $W : X \times X \times [0,1] \rightarrow X$ is said to be a convex structure on $X$ if for all $x, y \in X$ and $\lambda \in [0,1],$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

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holds for all \( u \in X \). The metric space \((X, d)\) together with a convex structure is called a **convex metric space** [13].

A subset \( K \) of a convex metric space \((X, d)\) is said to be a **convex set** [13] if \( W(x, y, \lambda) \in K \) for all \( x, y \in K \) and \( \lambda \in [0, 1] \).

A set \( K \) is said to be **\( p \)-starshaped** (see [6]) where \( p \in K \), provided \( W(x, p, \lambda) \in K \) for all \( x \in K \) and \( \lambda \in [0, 1] \) i.e. the segment

\[
[p, x] = \{ W(x, p, \lambda) : 0 \leq \lambda \leq 1 \}
\]

joining \( p \) to \( x \) is contained in \( K \) for all \( x \in K \). \( K \) is said to be **starshaped** if it is \( p \)-starshaped for some \( p \in K \).

Clearly, each convex set is starshaped but not conversely.

A convex metric space \((X, d)\) is said to satisfy **Property (I)** [6] if for all \( x, y, q \in X \) and \( \lambda \in [0, 1] \),

\[
d(W(x, q, \lambda), W(y, q, \lambda)) \leq \lambda d(x, y).
\]

A normed linear space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [6], [13]). Property (I) is always satisfied in a normed linear space.

For a non-empty subset \( M \) of a metric space \((X, d)\) and \( x \in X \), an element \( y \in M \) is said to be a **best approximant** of \( x \) in \( M \) or a **best \( M \)-approximant** to \( x \) if \( d(x, y) = d(x, M) \equiv \inf\{d(x, y) : y \in M\} \). The set of all such \( y \in M \) is denoted by \( P_M(x) \).

For a convex subset \( M \) of a convex metric space \((X, d)\), a mapping \( g : M \to X \) is said to be **affine** if for all \( x, y \in M \), \( g(W(x, y, \lambda)) = W(gx, gy, \lambda) \) for all \( \lambda \in [0, 1] \). \( g \) is said to be **affine with respect to \( p \in M \)** if \( g(W(x, p, \lambda)) = W(gx, gp, \lambda) \) for all \( x \in M \) and \( \lambda \in [0, 1] \).

Suppose \( M \) is a nonempty subset of a metric space \((X, d)\) and \( S, T \) are self mappings of \( M \). A point \( x \in M \) is a **common fixed (coincidence) point** of \( S \) and \( T \) if \( x = Sx = Tx(Sx = Tx) \). The set of fixed points (respectively, coincidence points) of \( S \) and \( T \) is denoted by \( F(S, T) \) (respectively, \( C(S, T) \)). The mappings \( T, S : M \to M \) are said to be

(i) **commuting** on \( M \) if \( STx = TSx \) for all \( x \in M \);
(ii) **\( R \)-weakly commuting** on \( M \) if there exists a real number \( R > 0 \) such that \( d(TSx, STx) \leq Rd(Tx, Sx) \) for all \( x \in M \).
(iii) **weakly compatible** if they commute at their coincidence points, i.e., if \( STx = TSx \) whenever \( Sx = Tx \);
(iv) **asymptotically \( S \)-nonexpansive** if there exists a sequence \( \{k_n\} \) of real numbers in \([1, \infty)\) with \( k_n \geq k_{n+1}, k_n \to 1 \) as \( n \to \infty \) such that \( d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy) \), for all \( x, y \in M \).
If $S$ = identity mapping in (iv), then $T$ is said to be an \textit{asymptotically nonexpansive mapping} and further, if $k_n = 1$ for all $n \in \mathbb{N}$, then $T$ is known as \textit{nonexpansive}.

$T$ is said to be \textit{uniformly asymptotically regular} on $M$ if, for each $\epsilon > 0$, there exists a positive integer $N$ such that $d(T^n(x), T^n(y)) < \epsilon$ for all $n \geq N$ and for all $x, y \in M$.

Suppose $(X, d)$ is a convex metric space, $M$ a $q$-starshaped subset with $q \in F(S) \cap M$ and is both $T$- and $S$-invariant. Then $T$ and $S$ are called

(i) \textit{$R$-subweakly commuting} (see [11]) on $M$ if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R \text{dist}(Sx, W(Tx, q, k), k \in [0, 1]$;

(ii) \textit{$C_q$-commuting} [1] if $STx = TSx$ for all $x \in C_q(S, T)$, where $C_q(S, T) = \{ C(S, T_k) : 0 \leq k \leq 1 \}$ and $T_k x = \{ W(Tx, q, k) : 0 \leq k \leq 1 \}$.

(iii) \textit{generalized asymptotically $S$-nonexpansive} (see [12]) if there exists a sequence $\{ k_n \}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that

$$d(T^n x, T^n y) \leq k_n \text{max} \{ d(Sx, Sy), \text{dist}(Sx, [T^n x, q]), \text{dist}(Sy, [T^n y, q]), \frac{1}{2} \text{dist}(Sx, [T^n y, q]) + \text{dist}(Sy, [T^n x, q]) \}$$

for all $x, y \in M$.

Clearly, $C_q$-commuting maps are weakly compatible. However, converse is not true.

\textbf{Example 2.1} ([1]). Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = x^2$ for all $x \neq 2$ and $T2 = 1$; and $Sx = 2x$ for all $x \in M$. Then $M$ is $q$-starshaped with $q = 0$, $C(T, S) = \{ 0 \}$ and $C_q(T, S) = \{ 0 \} \cup [2, \infty)$. Moreover, $T$ and $S$ are weakly compatible but not $C_q$-commuting.

Clearly, $R$-subweakly commuting mappings are $C_q$-commuting but converse does not hold.

\textbf{Example 2.2} ([1]). Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [0, \infty)$. Define $T, S : M \to M$ by $Tx = \frac{x}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$; and $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x \geq 1$. Then $M$ is $q$-starshaped with $q = 1$, and $C_q(T, S) = [1, \infty)$. Moreover $S$ and $T$ are $C_q$-commuting but not $R$-subweakly commuting.

If $T$ and $S$ are $C_q$-commuting on $M$, then $ST^n x = T^n Sx$ for all $x \in C_q(S, T^n)$, where $C_q(S, T^n) = \{ C(S, T_k) : 0 \leq k_n \leq 1 \}$ and $T_k x = \{ W(Tx, q, k_n) : 0 \leq k_n \leq 1 \}$.

\textbf{Example 2.3} ([12]). Let $X = \mathbb{R}$ be endowed with the usual metric and $M = [1, \infty)$. Then $M$ is $q$-starshaped with $q = 1$. Define $T, S : M \to M$ by $Tx = 2x^2 - 1$ and $Sx = 4x - 3$ for all $x \in M$. Then $C_q(T, S) = [1, \infty)$ and the pair is $C_q$-commuting on $M$. Now $C_q(T, S^n) = [1, \infty)$ for all $n \geq 1$. It is easy to verify that $TS^n x = S^n Tx$ for all $x \in c_q(T, S^n)$ and for each $n$. 
3. Main Results

To prove the main results, we need the following lemma. For normed linear spaces this lemma was proved in [12] and the proof can be easily extended to metric spaces.

**Lemma 3.1.** Let $M$ be a nonempty closed subset of a metric space $(X, d)$. Let $f, T : M \to M$ be self mappings, $q \in F(f)$ and $T(M\{\{q\}) \subset f(M\{\{q\})$. Suppose there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq k \max \left\{ d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(fy, Tx)) \right\}$$

(3.1)

for all $x, y \in M$. Further, if $T$ is continuous, $\text{cl} \ [T(M\{\{q\})]$ is complete, $f$ and $T$ are weakly compatible on $M\{\{q\}$, then $F(f) \cap F(T)$ is singleton.

The following theorem extends and generalizes the corresponding results of Al-Thagafi and Shahzad ([1]-Theorem 2.2), Hussain and Rhoades ([7]-Theorem 2.2) and of Vijayaraju and Hemavathy ([12]-Theorem 3.1).

**Theorem 3.2.** Let $M$ be a nonempty subset of a convex metric space $(X, d)$ with Property (I) and $T$ and $f$ are self mappings of $M$. Suppose that $M$ is $q$-starshaped with $q \in F(f)$ and $f$ is continuous and affine with respect to $q$. If $T$ and $f$ are $C_q$-commuting on $M\{\{q\}$, $\text{cl} \ [T(M\{\{q\})]$ is compact, $\text{cl} \ [T(M)] \subset f(M\{\{q\})$, and $T$ is continuous, uniformly asymptotically regular and generalized asymptotically $f$-nonexpansive i.e. it satisfies,

$$d(T^n x, T^n y) \leq \mu_n \max \{d(fx, fy), \text{dist}(fx, [T^n x, q]), \text{dist}(fy, [T^n y, q]),$$

$$\frac{1}{2}[\text{dist}(fx, [T^n y, q]) + \text{dist}(fy, [T^n x, q])]\}$$

(3.2)

for all $x, y \in M$, where $\mu_n$ is a sequence of real numbers with $\mu_n \geq 1$, and $\lim \mu_n = 1$, then $F(T) \cap F(f) \neq \emptyset$.

**Proof.** Let $(\lambda_n)$ be a sequence of real numbers such that $0 \leq \lambda_n < 1$ and $\lambda_n \to 1$. Take $k_n = \frac{\lambda_n}{\mu_n}$, then $(k_n)$ is a sequence of real numbers such that $0 \leq k_n < 1$, and $k_n \to 1$. Define $T_n$ as $T_n x = W[T^n x, q, k_n]$ for all $x \in M$ and for each $n \geq 1$. As $M$ is $q$-starshaped, $f$ is affine with respect to $q$ and $\text{cl} \ [T(M\{\{q\})] \subset f(M\{\{q\})$, $T_n$ is a self mapping of $M$ and $\text{cl}[T_n(M\{\{q\})] \subset f(M\{\{q\})$ for each $n$. Since $T$ and $f$ are $C_q$-commuting, $f T^n x = T^n f x$ for all $x \in C_q(f, T^n)$. As $f$ is affine with respect to $q$, it follows that for each $x \in C_q(f, T)$, $f T_n x = f(W[T^n x, q, k_n]) = W[f T^n x, f q, k_n] = W[T^n f x, f q, k_n] = T_n f x$. Thus $f T_n x = T_n f x$ for each $x \in C(f, T_n) \subset C_q(f, T^n)$. Hence the pair $f$ and $T_n$ are weakly compatible for all $n$. Further, we have

$$d(T_n x, T_n y) = d(W[T^n x, q, k_n], W[T^n y, q, k_n])$$
to the set of best approximation. For normed linear spaces, the following result is given in this, we need the following result.

Let $T$ and $f$ be self mappings of a nonempty subset $M$ of a convex metric space $X$. Suppose that $M$ is $q$-starshaped with $q \in F(f)$. If $T$ is continuous, $f$ is continuous and affine with respect to $q$, $cl\{T(M)\}$ is compact and $cl\{T(M)\} \subseteq f(M)\setminus\{q\}$. If $T$ and $f$ are $C_q$-commuting on $M\setminus\{q\}$ and $T$ is uniformly asymptotically regular and asymptotically non-expansive with sequence $\{\mu_n\}$, where $\{\mu_n\}$ is a sequence of real numbers with $\mu_n \geq 1$ and $\mu_n \to 1$, then $F(T) \cap F(f) \neq \emptyset$.

We shall now give an application of Theorem 3.2 to the set of best approximation. For this, we need the following result.
Proposition 3.4. If $M$ is a subset of a convex metric space $(X, d)$, $u \in X \setminus M$ and $y \in P_M(u)$, then the line segment $\{W(y, u, \lambda) : 0 < \lambda < 1\}$ and the set $M$ are disjoint.

Proof. Since $y \in P_M(u)$, consider

$$d(u, W(y, u, \lambda)) \leq \lambda d(u, y)$$

$$< d(u, M), \text{ for every } 0 < \lambda < 1.$$ 

This implies that $W(y, u, \lambda) \notin M$ for any $\lambda$, $0 < \lambda < 1$. Therefore the line segment $\{W(y, u, \lambda) : 0 < \lambda < 1\}$ and the set $M$ are disjoint. \qed

Taking $G = M \setminus q$ for some $q \in M$, we have the following result.

Theorem 3.5. Let $M$ be a nonempty subset of a convex metric space $(X, d)$ with Property (I) and $T$ and $f$ are self mappings of $X$ such that $T(\partial G \cap G) \subseteq G$ and $u \in F(T) \cap F(f)$ for some $u \in X \setminus M$, where $\partial G$ denotes boundary of $G$. Suppose that $P_G(u)$ is closed and $q$-starshaped and $f$ is affine with respect to $q \in F(f)$ with $f(P_G(u)) = P_G(u)$. If $T$ and $f$ are continuous, $C_q$-commuting on $P_G(u) \cup \{u\}$ satisfying $d(Tx, Tu) \leq d(f x, f u)$, $cl \ T[P_G(u)$ is compact, and $T$ is uniformly asymptotically regular and generalized asymptotically $f$-nonexpansive mapping for all $x, y \in P_G(u)$, then $P_G(u) \cap F(T) \cap F(f) \neq \emptyset$.

Proof. Let $x \in D = P_G(u)$, then for any $k \in (0, 1]$, we have

$$d(W(u, x, k), u) \leq kd(u, u) + (1 - k)d(x, u) = (1 - k)d(x, u) < \text{dist}(u, G).$$

It follows from Proposition 3.4 that the open line segment $\{W(u, x, \lambda) : 0 < \lambda < 1\}$ and the set $G$ are disjoint. Thus $x$ is not in the interior of $G$ and so $x \in \partial G \cap G$. Since $T(\partial G \cap G) \subseteq G$, $Tx$ must be in $G$. Also, $f(P_G(u)) = P_G(u)$, $f x \in P_G(u)$, $u \in F(T) \cap F(f)$, we have

$$d(Tx, u) = d(Tx, Tu) \leq d(f x, f u) = d(f x, u) \leq \text{dist}(u, G).$$

This implies that $Tx \in P_G(u)$. Consequently, $D = P_G(u)$ is $T$ invariant. Hence by Theorem 3.2, there exists $z \in P_G(u)$ such that $P_G(u) \cap F(T) \cap F(f) \neq \emptyset$. \qed

Remark 2. Theorem 3.5 extends the corresponding results of [8], [9], [10] and [11] to generalized asymptotically $f$-nonexpansive mappings.

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