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ANTINORMAL COMPOSITION OPERATORS ON ℓ^2

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Abstract. A bounded linear operator T on a Hilbert space H is called antinormal if the distance of T from the set of all normal operators is equal to norm of T. In this paper, we give a complete characterization of antinormal composition operators on ℓ^2, where ℓ^2 is the Hilbert space of all square summable sequences of complex numbers under standard inner product on it.

1. Introduction

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H. For an operator $T \in B(H)$, the distance $d(T, \boldsymbol{M})$ of an operator T from $\boldsymbol{M} \subset B(H)$ is given by $d(T, \boldsymbol{M}) = \inf\{\|T - M\| : M \in \boldsymbol{M}\}$.

It is an important problem to compute the distance of an operator from the set of all normal operators. For details and some open problem, see [3, p.155]. An operator T is called antinormal if $d(T, \boldsymbol{N}) = \|T\|$ where $\boldsymbol{N}$ is the set of all normal operators in $B(H)$. This definition was first given by Holmes [8] in 1974. Fruther study of antinormal operators has been done by Rogers, Izumino and Elst, see [1], [2], [5], [9] and [10].

For an operator T in $B(H)$ index of T [1] is given by

$$\text{index } T = \begin{cases} 0 & \text{if dim ker } T = \text{dim ker } T^* \\ \text{dim ker } T - \text{dim ker } T^* & \text{if dim ker } T < N_0 \text{ and} \\ & \quad \text{dim ker } T^* < N_0. \\ \text{dim ket } T & \text{if dim ker } T \geq N_0 \text{ and} \\ & \text{dim ker } T^* < \text{dim ker } T \\ -\text{dim ket } T^* & \text{if dim ker } T^* \geq N_0 \text{ and} \\ & \text{dim ker } T < \text{dim ker } T^* \end{cases}$$

Observe that index $T = -$index T^* for all $T \in B(H)$.

Minimum modulus $m(T)$ of an operator T is defined by [7] $m(T) = \inf\{\lambda \geq 0 : \lambda \in \sigma(|T|)\}$. It is easy to see that $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$ also. *Essential minimum modulus* m_e of an operator T is defined by [7] $m_e(T) = \inf\{\lambda \geq 0 : \lambda \in \sigma_e(|T|)\} = \inf\{\lambda \geq 0 : \lambda I - |T| \text{ is not Fredholm}\}$. Here $|T|$ denotes $\sqrt{T^*T}$ and $\sigma_e(T)$ denotes the essential spectrum of T.

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Now we give some results for an operator T on a separable Hilbert space H which are useful in our context.

Theorem 1.1.([10]) *Let $T \in B(H)$.*
(i) *If index $T = 0$, then $d(T, \boldsymbol{U}) = \max\{\|T\| - 1, 1 - m(T)\}$*
(ii) *If index $T < 0$, then $d(T, \boldsymbol{U}) = \max\{\|T\| - 1, 1 + m_e(T)\}$*
where $\boldsymbol{U}$ is the set of all unitary operators in $B(H)$.

Sinde $d(T, \boldsymbol{U}) = d(T^*, \boldsymbol{U})$, we consider T^* when index $T > 0$.

Theorem 1.2.([9]) *Let $T \in B(H)$.*
(i) *If index $T = 0$, then $d(T, \boldsymbol{N}) \le (\|T\| - m(T))/2$*
(ii) *If index $T < 0$, then $m_e(T) \le d(T, \boldsymbol{N}) \le (\|T\| + m_e(T))/2$*

Remark 1.3. If index $T = 0$, then T cannot be antinormal.

Theorem 1.4.([9]) *Let $T \in B(H)$ with index $T < 0$, then the following conditions are equivalent.*
(i) *T is antinormal*
(ii) *$m_e(T) = \|T\|$*
(iii) *$d(T, \boldsymbol{U}) = 1 + \|T\|$.*

Remark 1.5. T is antinormal iff T^* is antinormal because $d(T, \boldsymbol{N}) = d(T^*, N)$.

The composition operator C_ϕ on ℓ^2 is defined as [10] $C_\phi(f) = f o\phi$, where ϕ is the inducing function on $\boldsymbol{N}$ into itself satisfying
(i) $A_n = \phi^{-1}(n)$ is finite for each $n \in \boldsymbol{N}$.
(ii) $\{\overline{\overline{A_n}} : n \in \boldsymbol{N}\}$ is bounded where $\overline{\overline{A_n}}$ denotes the number of elements in A_n.

2. Main Result

Now we state some results on composition operator on ℓ^2, which will be useful in proof of our main result, Theorem 2.3 of this section.

Theorem 2.1.([11]) *Let C_ϕ be a composition operator on ℓ^2 then*
(i) *ϕ is one-one iff C_ϕ is noto iff C_ϕ^* is one-one.*
(ii) *ϕ is onto iff C_ϕ is one-one iff C_ϕ^* is onto.*
(iii) *ϕ is one-one and onto iff C_ϕ is normal iff C_ϕ is invertible.*

Proposition 2.2. *Let C_ϕ be a composition operator on ℓ^2. Then the range of the operator αI - $C_\phi^* C_\phi$ is colsed for each $\alpha \in C$.*

Proof. Let $f = \Sigma f(j)\chi_j$ be an element of ℓ^2, where χ_j is the characteristic function of $\{j\}$.
We have

$$C_\phi^* C_\phi(f) = C_\phi^* C_\phi\Big(\sum_{j=1}^{\infty} f(j)\chi_j\Big)$$

$$= C_\phi^*\Big(\sum_{j=1}^{\infty} f(j)\chi_{\phi^{-1}(j)}\Big)$$
$$= \sum_{j=1}^{\infty} \overline{\overline{A_j}} f(j)\chi_j.$$

Now $(\alpha I\text{-}C_\phi^* C_\phi)(f) = \sum_{j=1}^{\infty}(\alpha - \overline{\overline{A_j}}) f(j)\chi_j$.

Thus $\alpha I\text{-}C_\phi^* C_\phi$ is a diagonal operator and the spectrum of $\alpha I\text{-}C_\phi^* C_\phi$ consists of only a finite number of distinct points. Hence zero is not a limit point of spectrum of $\alpha I\text{-}C_\phi^* C_\phi$. Therefore range of $\alpha I\text{-}C_\phi^* C_\phi$ is closed because the range of a normal operator is closed iff 0 is not a limit point of its spectrum [3].

Now we characterize antinormal composition operators on ℓ^2.

Theorem 2.3. *Let C_ϕ be a composition operator on ℓ^2.*

(i) *If ϕ is one-one and onto then C_ϕ is not antinormal.*
(ii) *If ϕ is one-one but not onto then C_ϕ is antinormal.*
(iii) *ϕ is onto but not one-one then C_ϕ is antinormal iff $\overline{\overline{A_n}} = \|C_\phi\|^2$ for all but finitely many $n \in \boldsymbol{N}$.*
(iv) *Suppose ϕ is neither one-one nor onto.*

(a) If index $C_\phi < 0$, then C_ϕ is antinormal iff $\overline{\overline{A_n}} = \|C_\phi\|^2$ for all but finitely many $n \in \boldsymbol{N}$.
(b) if index $C_\phi \geq 0$ then C_ϕ is not antinormal.

Proof.

(i) If ϕ is one-one and onto, then by Theorem 2.1 (iii) C_ϕ is a non-zero normal operator and hence by definition it follows that C_ϕ is not antinormal.
(ii) If ϕ is one-one but not onto, we shall show that C_ϕ is antinormal. Since ϕ is one-one but not onto, dim ker $C_\phi^* = 0$ and dim ket $C_\phi \neq 0$. Therefore index $C_\phi^* < 0$.
Let $f \in \ell^2$,

$$\text{then } C_\phi C_\phi^*(f) = C_\phi\Big(C_\phi^* \sum_{j=1}^{\infty} f(j)\chi_j\Big)$$
$$= C_\phi\Big(\sum_{j=1}^{\infty} f(j)\chi_{\phi(j)}\Big)$$
$$= \sum_{j=1}^{\infty} f(j)\chi_{\phi^{-1}(\phi(j))}$$
$$= \sum_{j=1}^{\infty} f(j)\chi_j \qquad (\text{because } \phi \text{ is } 1-1)$$
$$= I(f).$$

Therefore $C_\phi C_\phi^* = I$. As dim ker $C_\phi \neq 0$, C_ϕ is a non-surjective isometry. Then from Theorem 8 and Theorem 9 in [1] it follows that C_ϕ^* is antinormal and by Remark 1.5, C_ϕ is antinormal.

(iii) If ϕ is onto but not one-one, then index $C_\phi < 0$ by Theorem 2.1. Suppose $\overline{\overline{A_n}} = \|C_\phi\|^2$ for all but finitely many $n \in \boldsymbol{N}$. Now to prove that C_ϕ is antinormal, we shall show that $m_e(C_\phi) = \|C_\phi\|$. We note that range $(kI\text{-}C_\phi^* C_\phi)$ is closed in particular for $0 \le k < \|C_\phi\|^2$ by Proposition 2.2. The set $(n \in \boldsymbol{N} : \overline{\overline{A_n}} = k\}$ is a finite subset of $\boldsymbol{N}$ for each $0 \le k < \|C_\phi\|^2$. For $f \in \ell^2$, $(kI\text{-}C_\phi^* C_\phi)(f) = \sum_{j=j}^{\infty}(k - \overline{\overline{A_j}}) f(j)\chi_j$. Therefore dim ker $(kI\text{-}C_\phi^*)$ = dim ket $(kI\text{-}C_\phi^* C_\phi)$ is finite for each $0 \le k < \|C_\phi\|^2$. Hence $kI - C_\phi^* C_\phi$ is Fredholm for $0 \le k < \|C_\phi\|^2$. Thus $k \not\in \sigma_e(|C_\phi|)$ for $0 \le k < \|C_\phi\|$. Now dim ker $(\|C_\phi\|^2 I - C_\phi^* C_\phi)$ is infinite because $\overline{\overline{A_n}} = \|C_\phi\|^2$ for all but finitely many $n \in \boldsymbol{N}$. Hence $\|C_\phi\| \in \sigma_e(|C_\phi|)$. Therefore $m_e(C_\phi) = \inf\{\lambda \ge 0 : \lambda \in \sigma_e(|C_\phi|)\} = \|C_\phi\|$. Thus C_ϕ is antinormal.

Conversely suppose that the set $\{n \in \boldsymbol{N} : \overline{\overline{A_n}} \ne \|C_\phi\|^2\}$ is infinite. It is easy to see that there exists a non-negative integer $k_0 < \|C_\phi\|^2$ such that $\{n \in \boldsymbol{N} : \overline{\overline{A_n}} = k_0\}$ is infinite. Therefore dime ker $(k_0 I - C_\phi^* C_\phi)$ is infinite and hence $(k_0 I - C_\phi^* C_\phi)$ is not Fredholm. Thus $\sqrt{k_0} \in \sigma_e(|C_\phi|)$ and $m_e(C_\phi) \le \sqrt{k_0} < \|C_\phi\|$. Hence C_ϕ is not antinormal.

(iv) Suppose that ϕ is neither one-one nor onto.

(a) If index $C_\phi < 0$ then the proof is similar to that of case (iii).

(b) If index $C_\phi > 0$ then index $C_\phi^* < 0$. We shall show that C_ϕ^* is not antinormal. Since ℓ^2 is separable, index $C_\phi^* < 0$ implies that dim ker C_ϕ^* is finite. Since dim ker $C_\phi^* = \sum_{\overline{\overline{A_n}} > 1} (\overline{\overline{A}}_n - 1)$ is finite, it is easy to see that $\overline{\overline{A}}_n = 1$ for infinitely many $n \in \boldsymbol{N}$. Now $(I - C_\phi C_\phi^*)(f) = \sum_{j=1}^{\infty} f(j)\chi_j - \sum_{j=1}^{\infty} f(j)\chi_{\phi^{-1}(\phi(j))}$. Since $\overline{\overline{A}}_n = 1$ for infinitily many $n \in \boldsymbol{N}$, and $n \in \phi^{-1}(\phi(n))$ for each $n \in \boldsymbol{N}$, Ker $(I - C_\phi C_\phi^*)$ is infinite dimensional. Therefore $I - C_\phi C_\phi^*$ is not Fredholm. Thus $1 \in \sigma_e(|C_\phi^*|)$ and $m_e(C_\phi^*) \le 1 < \|C_\phi\| = \|C_\phi^*\|$. Hence C_ϕ^* is not antinormal.

If index $C_\phi = 0$ then C_ϕ cannot be antinormal by Remark 1.3.

3. Examples

1. The function ϕ on $\boldsymbol{N}$ into itself defined by $\phi(n) = n + 1$ is one-one but not onto. The composition operator C_ϕ induced by ϕ is antinormal by case (ii). This is the unilateral shift.
2. Define the function ϕ on N into itself by

$$\phi(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ \dfrac{n+3}{2} & \text{if } n > 2 \text{ and } n \text{ is odd} \\ \dfrac{n}{2} + 1 & \text{if } n > 2 \text{ and } n \text{ is even} \end{cases}.$$

Here ϕ is onto but not one-one. The composition operator C_ϕ is antinormal by case (iii) because $\overline{\overline{A}}_n = 2 = \|C_\phi\|^2$ for all n in $\boldsymbol{N}$ except $n = 1$ and 2.

3. Define the function ϕ on $\boldsymbol{N}$ into itself by

$$\phi(n) = \begin{cases} 1 & \text{if } n = 1,2 \\ 2 & \text{if } n = 3 \\ m+5 & \text{if } n = 3m+1, 3m+2, 3m+3 \end{cases}.$$

Here ϕ is neither one-one nor onto and $\dim\ker C_\phi = \overline{\overline{N - \phi(N)}} = 3$, $\dim\ker C_\phi^* = \sum_{\overline{\overline{A}}_n > 1} (\overline{\overline{A}}_n - 1) = N_0$. Thus index $C_\phi < 0$. Therefore C_ϕ is antinormal by case iv (a) because $\overline{\overline{A}}_n = \|C_\phi\|^2 = 3$ for all n in $\boldsymbol{N}$ except $1 \le n \le 3$.

4. Define the function ϕ on $\boldsymbol{N}$ into itself by

$$\phi(n) = \begin{cases} 1 & \text{if } n = 1,2 \\ n+2 & \text{if } n \ge 3 \end{cases}.$$

Here ϕ is neither one-one nor onto and index $C_\phi = 2 > 0$. Therefore C_ϕ is not antinormal by case iv (b).

5. Define the function ϕ on N into itself by

$$\phi(n) = \begin{cases} 1 & \text{if } n = 1,2 \\ n & \text{if } n > 2 \end{cases}.$$

Here ϕ is neither one-one nor onto and index $C_\phi = 0$. Therefore C_ϕ is not antinormal by case iv (b).

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References

[1] AFM ter Elst, *Antinormal operators*, Acta. Sci. Math. **54**(1990), 151–158.

[2] AFM ter Elst, *Approximation by unitary operators*, Acta. Sci. Math. **54**(1990), 145–149.

[3] I. D. Berg and K. R. Davidson, *Almost commuting matrices and a quantitative version of the Brown, Douglas and Fillmore Theorem*, Acta. Math. **166**(1991), 121–161.

[4] John. B. Conway, A Course in Functional Analysis, second edition, Springer – Verlag, 1990.

[5] M. Fujii and R. Nakamoto, *Antinormal operators and theorems of Izzumino*, Math. Japon. **24**(1979), 41–44.

[6] P. R. Halmos, A Hilbert Space Problem Book, Princeton, New Jersey, Von Nostrand, 1967.

[7] D. A. Herrero, Approximation of Hilbert Space Operators, Vol. 1, second edition. Pitnam Research Notes in Mathematics Series. Longman Scientific and Technical, 1989.

[8] R. B. Holmes, *Best approximation by normal operators*, J. Approx. Theory **12**(1974), 412–417.

[9] Saichi Izumino, *Inequalities on normal and antinormal operaotrs*, Math. Japonica **23**(1978), 211–215.

[10] Donald D. Rogers, *Approximation by unitary and anti unitary operators*, Acta. Sci. Math. **39**(1977), 141–151.

[11] R. K. Singh and J. S. Manhas, Composition Operators on Function Spaces. North Holland, 1993.

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