TAMKANG JOURNAL OF MATHEMATICS
Volume 38, Number 1, 1-20, Spring 2007

ON A SYSTEM OF NONLINEAR WAVE EQUATIONS OF KIRCHHOFF TYPE WITH A STRONG DISSIPATION

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Abstract. The initial boundary value problem for systems of nonlinear wave equations of Kirchhoff type with strong dissipation in a bounded domain is considered. We prove the local existence of solutions by Banach fixed point theorem and blow-up of solutions by energy method. Some estimates for the life span of solutions are given.

1. Introduction

We consider the initial boundary value problem for the following nonlinear coupled wave equations of Kirchhoff type :

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u - \Delta u_t = f_1(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

$$v_{tt} - M\left(\|\nabla v\|_2^2\right)\Delta v - \Delta v_t = f_2(u,v) \quad \text{in } \Omega \times [0,\infty), \tag{1.2}$$

with initial conditions,

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{1.4}$$

and boundary conditions,

$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \tag{1.5}$$

$$v(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \tag{1.6}$$

where $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Let $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ be the Laplace operator, and $M(r)$ be a nonnegative locally Lipschitz function for $r \geq 0$ like $M(r) = a + br^{\gamma}$, with $a \geq 0$, $b > 0$, $a + b \geq 0$, $\gamma \geq 0$, and $f_i(u,v)$, $i = 1,2$, be a nonlinear function. We denote $\|\cdot\|_p$ to be L^p-norm.

The existence and nonexistence of solutions for a single wave equation of Kirchhoff type :

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + g(u_t) = f(u) \quad \text{in } \Omega \times [0,\infty), \tag{1.7}$$

Received May 20, 2004; revised December 2, 2004.
2000 *Mathematics Subject Classification.* 35L70, 35L15, 65M60.
Key words and phrases. Blow-up, life span, dissipation, nonlinear wave equations.

have been discussed by many authors and the references cited therein. The function g in (1.7) is considered in three different cases. For $g(u_t) = \delta u_t$, $\delta > 0$, the global existence and blow-up results can be found in [4, 10, 16, 22]; for $g(u_t) = -\Delta u_t$, some global existence and blow-up results are given in [8, 10, 14, 17, 18, 22]; for $g(u_t) = |u_t|^m u_t$, $m > 0$, the main results of existence and blow-up are in [2, 3, 13, 15, 22]. As a model it describes the nonlinear vibrations of an elastic string. More precisely, we have

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.8}$$

for $0 < x < L, t \geq 0$; where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. Kirchhoff [11] was the first one to study the oscillations of streched strings and plates, so that (1.8) is named the wave equation of Kirchhoff type. Moreover, (1.8) is called a degenerate equation when $p_0 = 0$ and nondegenerate one when $p_0 > 0$. For the system of wave equations with no dissipative terms, when $M = 1$ many authors have discussed the local, global existence and blow-up properties in [5, 6, 7, 12]. When M is not a constant, Park and Bae [19, 20] considered the system of wave equations with $f_1(u, v) = |u|^\beta u$, $f_2(u, v) = |v|^\beta v$, $\beta \geq 0$, and showed the global existence and asymptotic behavior of solutions under some restrictions on initial energy. Later, Benaissa and Messaoudi [3] studied blow-up properties for negative initial energy.

In this paper, we investigate the local existence, blow-up properties of solutions for some nonlinear coupled wave equations of Kirchhoff type (1.1)−(1.6) in a bounded domain Ω with more general $f_i(u, v)$, $i = 1, 2$. The paper is organized as follows. In section 2, we present the preliminaries and some lemmas. In section 3, we will show the existence of a unique local weak solution (u, v) of our problem (1.1)−(1.6) with $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ by applying the Banach contraction mapping principle. In section 4, we first define an energy function $E(t)$ by (4.1) and show that it is a nonincreasing function. Then by using the direct method [12], we obtain Theorem 4.4, which shows blow-up properties of solutions under some restrictions even for positive initial energy, that is, we prove that there exists a finite time $T^* > 0$ such that $\lim_{t \to T^{*-}} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx = \infty$. Estimates for the blow-up time T^* are also given in Remark 4.5.

2. Preliminaries

Let us begin by stating the following lemmas, which will be used later.

Lemma 2.1.(Sobolev-Poincaré)[17, p.154]) *Let* $0 < p \leq \frac{2N}{N-2m}$ $(0 < p < \infty$ *if* $N = 2m)$. *Then, the inequality*

$$\|v\|_p \leq c_* \left\| (-\Delta)^{\frac{m}{2}} v \right\|_2 \tag{2.1}$$

holds with some constant c_*.

Lemma 2.2. *Let* $\delta > 0$ *and* $B(t) \in C^2(0,\infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.2}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.3}$$

then

$$B'(t) > K_0$$

for $t > 0$, *where* K_0 *is a constant,* $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ *is the smallest root of the equation*

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Proof. see [12].

Lemma 2.3. *If* $J(t)$ *is a nonincreasing function on* $[t_0,\infty)$ *and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}}, \tag{2.4}$$

where $a > 0, b \in R$, *then there exists a finite time* T^* *such that*

$$\lim_{t\to T^{*-}} J(t) = 0$$

and the upper bound of T^* *is estimated respectively by the following cases:*
(i) *If* $b < 0$ *and* $J(t_0) < \min\{1, \sqrt{-a/b}\}$, *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) *If* $b = 0$, *then*

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}.$$

(iii) *If* $b > 0$, *then*

$$T^* \leq \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{a}{b})^{2+\frac{1}{\delta}}$.

Proof. see [12].

3. Local existence

In this section we shall discuss the local existence of solutions for (1.1)−(1.6) by method of Banach fixed point theorem under the following assumptions on $f_i(u,v)$, $i = 1,2$. In the sequal, for the sake of simplicity we will omit the dependence on t, when the meaning is clear.

(A1) $f_i : R^2 \to R$ is continuously differentiable such that for each $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have uf_1, $vf_2 \in L^1(\Omega)$, and $F(u,v) \in L^1(\Omega)$, where

$$F(u,v) = \int_0^u f_1(s,v)ds + \int_0^v f_2(0,s)ds.$$

(A2) $f_i(0,0) = 0$ and for any $\rho > 0$ there exists a constant $k(\rho) > 0$ such that

$$|f_i(u_1,v_1) - f_i(u_2,v_2)| \leq k(\rho)\left[(|u_1|^\alpha + |u_2|^\alpha)|u_1 - u_2| + \left(|v_1|^\beta + |v_2|^\beta\right)|v_1 - v_2|\right],$$

where $|u_i|$, $|v_i| \leq \rho$, for $u_i, v_i \in R$, $i = 1,2$, and $0 \leq \alpha \leq \frac{4}{N-4}$, $0 \leq \beta \leq \frac{4}{N-4}$.

(A3) $\frac{\partial f_1}{\partial v} = \frac{\partial f_2}{\partial u}$.

Note the function of the form $f_1(u,v) = u^{s-1}v^s + u^p$, $f_2(u,v) = v^{s-1}u^s + v^q$ satisfy the assumptions (A1)−(A3) where $1 < s, p, q \leq \frac{N}{N-4}$ for $N \geq 4$ or $s, p, q > 1$ for $N = 1,2,3$. Before proving the existence theorem for nonlinear equations, we will give the definition of weak solution of (1.1)−(1.6) and we need the existence result for a nonhomgeneous wave equation with a strong dissipation.

Definition. A function $w(t) = (u(t), v(t)) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $t \in [0,T)$, is called a weak solution of (1.1)−(1.6) if

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t\eta dx = -\int_\Omega M\left(\|\nabla u\|_2^2\right)\nabla u\nabla\eta dx - \int_\Omega \nabla u_t\nabla\eta dx + \int_\Omega f_1(u,v)\eta dx, \\ \frac{d}{dt}\int_\Omega v_t\eta dx = -\int_\Omega M\left(\|\nabla v\|_2^2\right)\nabla v\nabla\eta dx - \int_\Omega \nabla v_t\nabla\eta dx + \int_\Omega f_2(u,v)\eta dx, \end{cases} \tag{3.1}$$

holds for any $\eta \in H_0^1(\Omega)$.

Theorem 3.1. *Let $m(t)$ be a nonnegative Lipschitz function and $f(t)$ be a Lipschitz function on $[0,T]$, $T > 0$. If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then there exists a unique solution u satisfying*

$$u(t) \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)),$$
$$u'(t) \in C^0([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega)),$$

and

$$u'' - m(t)\Delta u - \Delta u' = f(t) \text{ in } \Omega \times [0,T],$$
$$u(0) = u_0,\ u'(0) = u_1,\ x \in \Omega,$$
$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0,$$

here $u' = \frac{\partial u}{\partial t}$.

Proof. See [17, Prop. 2.2].

Theorem 3.2. *Assume* (A2) *holds and* $M(r)$ *is a nonnegative locally Lipschitz function for* $r \geq 0$ *with the Lipschitz constant* L. *If* $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *then there exist a unique local weak solutions* (u, v) *of* (1.1)−(1.6) *satisfying*

$$u(t), v(t) \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)),$$

and

$$u'(t), v'(t) \in C^0([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega)), \text{ for } T > 0.$$

Moreover, at least one of the following statements hold :
(i) $T = \infty$
(ii) $e(u(t), v(t)) \equiv \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|v_t\|_2^2 + \|\Delta v\|_2^2 \to \infty$ as $t \to T^-$.

Proof. We set $w(t) = (u(t), v(t))$, and define the following two-parameter space :

$$X_{T,R} = \left\{ \begin{array}{c} w(t) \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)), w_t(t) \in C^0([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega)) : \\ e(u(t), v(t)) \leq R^2, \text{ with } w(0) = (u_0, v_0), \ w_t(0) = (u_1, v_1). \end{array} \right\},$$

for $T > 0$, $R > 0$. Then $X_{T,R}$ is a complete metric space with the distance

$$d(y,z) = \sup_{0 \leq t \leq T} \left\{ \|(\mu - \varphi)_t\|_2^2 + \|\Delta\mu - \Delta\varphi\|_2^2 + \|(\xi - \psi)_t\|_2^2 + \|\Delta\xi - \Delta\psi\|_2^2 \right\}^{\frac{1}{2}}, \quad (3.2)$$

where $y(t) = (\mu(t), \xi(t))$, $z(t) = (\varphi(t), \psi(t)) \in X_{T,R}$.

Given $\overline{w}(t) = (\overline{u}(t), \overline{v}(t)) \in X_{T,R}$, we consider the linear system

$$u_{tt} - M\left(\|\nabla \overline{u}\|_2^2\right) \Delta u - \Delta u_t = f_1(\overline{u}, \overline{v}) \quad \text{in } \Omega \times [0,T), \quad (3.3)$$

$$v_{tt} - M\left(\|\nabla \overline{v}\|_2^2\right) \Delta v - \Delta v_t = f_2(\overline{u}, \overline{v}) \quad \text{in } \Omega \times [0,T), \quad (3.4)$$

with initial conditions,

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (3.5)$$

$$v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \quad x \in \Omega, \quad (3.6)$$

and boundary conditions,

$$u(x,t) = 0, \ x \in \partial\Omega, \ t > 0, \quad (3.7)$$

$$v(x,t) = 0, \ x \in \partial\Omega, \ t > 0. \quad (3.8)$$

By Theorem 3.1, there exists a unique solution $w(t) = (u(t), v(t))$ of (3.3)−(3.8). We define the nonlinear mapping $S\overline{w} = w$, and then, we will show that there exist $T > 0$ and $R > 0$ such that

(i) $S : X_{T,R} \to X_{T,R}$,
(ii) S is a contraction mapping in $X_{T,R}$ with respect to the metric $d(\cdot,\cdot)$ defined in (3.2).

Indeed, multiplying (3.3) by $2u_t$, integrating it over Ω, and then by Divergence theorem, we get

$$\frac{d}{dt}\left\{\|u_t\|_2^2 + M\left(\|\nabla\overline{u}\|_2^2\right)\|\nabla u\|_2^2\right\} + 2\|\nabla u_t\|_2^2 = I_{u1} + I_{u2}, \tag{3.9}$$

where

$$I_{u1} = \left(\frac{d}{dt}M\left(\|\nabla\overline{u}\|_2^2\right)\right)\|\nabla u\|_2^2, \tag{3.10}$$

$$I_{u2} = \int_\Omega 2f_1(\overline{u},\overline{v})u_t dx. \tag{3.11}$$

Similarly, we also have

$$\frac{d}{dt}\left\{\|v_t\|_2^2 + M\left(\|\nabla\overline{v}\|_2^2\right)\|\nabla v\|_2^2\right\} + 2\|\nabla v_t\|_2^2 = I_{v1} + I_{v2}, \tag{3.12}$$

where

$$I_{v1} = \left(\frac{d}{dt}M\left(\|\nabla\overline{v}\|_2^2\right)\right)\|\nabla v\|_2^2, \tag{3.13}$$

$$I_{v2} = \int_\Omega 2f_2(\overline{u},\overline{v})v_t dx. \tag{3.14}$$

Note that from Lemma 2.1, Divergence theorem, and $\overline{w} \in X_{T,R}$, we have

$$\begin{aligned} |I_{u1}| &\le 2L\|\Delta\overline{u}\|_2\|\overline{u_t}\|_2\|\nabla u\|_2^2 \\ &\le 2Lc_*^2R^2\|\Delta u\|_2^2 \\ &\le c_0LR^2e(u,v), \end{aligned} \tag{3.15}$$

where $c_0 = 2c_*^2$. In the same way, we get

$$|I_{v1}| \le c_0LR^2e(u,v). \tag{3.16}$$

By (3.11), (A2), and Lemma 2.1, we have

$$\begin{aligned} |I_{u2}| &\le 2k\int_\Omega\left(|\overline{u}|^{\alpha+1} + |\overline{v}|^{\beta+1}\right)|u_t|\,dx \\ &\le 2k\left[\left(c_*\|\Delta\overline{u}\|_2\right)^{\alpha+1} + \left(c_*\|\Delta\overline{v}\|_2\right)^{\beta+1}\right]\|u_t\|_2 \\ &\le c_1\left(R^{\alpha+1} + R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}, \end{aligned} \tag{3.17}$$

where $c_1 = 2k\max\left(c_*^{\alpha+1}, c_*^{\beta+1}\right)$. Similarly, we have

$$|I_{v2}| \le c_1\left(R^{\alpha+1} + R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}. \tag{3.18}$$

Combining (3.9) and (3.12) together, and by (3.15)−(3.18), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\|u_t\|_2^2+M\left(\|\nabla\overline{u}\|_2^2\right)\|\nabla u\|_2^2+\|v_t\|_2^2+M\left(\|\nabla\overline{v}\|_2^2\right)\|\nabla v\|_2^2\right\}\\
&\quad+2\|\nabla u_t\|_2^2+2\|\nabla v_t\|_2^2\\
&\quad\le 2c_0LR^2e(u,v)+2c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}. \qquad (3.19)
\end{aligned}$$

On the other hand, multiplying (3.3) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u dx\right\}+2M\left(\|\nabla\overline{u}\|_2^2\right)\|\Delta u\|_2^2\\
&\quad=2\|\nabla u_t\|_2^2-\int_\Omega 2f_1(\overline{u},\overline{v})\Delta u dx\\
&\quad\le 2\|\nabla u_t\|_2^2+c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}, \qquad (3.20)
\end{aligned}$$

the last inequality in (3.20) is obtained by following the arguments in (3.17). Similarly, we also have

$$\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta v\|_2^2-2\int_\Omega v_t\Delta v dx\right\}+2M\left(\|\nabla\overline{v}\|_2^2\right)\|\Delta v\|_2^2\\
&\quad\le 2\|\nabla v_t\|_2^2+c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(v(t))^{\frac{1}{2}}, \qquad (3.21)
\end{aligned}$$

Now, combining (3.20) and (3.21), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u dx+\|\Delta v\|_2^2-2\int_\Omega v_t\Delta v dx\right\}\\
&\quad+2M\left(\|\nabla\overline{u}\|_2^2\right)\|\Delta u\|_2^2+2M\left(\|\nabla\overline{v}\|_2^2\right)\|\Delta v\|_2^2\\
&\quad\le 2\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+2c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}. \qquad (3.22)
\end{aligned}$$

Multiplying (3.22) by ε, $0<\varepsilon\le 1$, and adding (3.19) together, we obtain

$$\begin{aligned}
&\frac{d}{dt}e^*_{\overline{u},\overline{v}}(u,v)+2(1-\varepsilon)\left[\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right]\\
&\quad\le 2c_0LR^2e(u,v)+2(1+\varepsilon)c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(u,v)^{\frac{1}{2}}, \qquad (3.23)
\end{aligned}$$

where

$$\begin{aligned}
e^*_{\overline{u},\overline{v}}(u,v)&=\|u_t\|_2^2+\|v_t\|_2^2+M\left(\|\nabla\overline{u}\|_2^2\right)\|\nabla u\|_2^2+M\left(\|\nabla\overline{v}\|_2^2\right)\|\nabla v\|_2^2\\
&\quad-2\varepsilon\left(\int_\Omega u_t\Delta u dx-\int_\Omega v_t\Delta v dx\right)+\varepsilon\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right). \qquad (3.24)
\end{aligned}$$

By Young′s inequality, we get $\left|2\varepsilon\int_\Omega u_t\Delta u dx\right|\le 2\varepsilon\|u_t\|_2^2+\frac{\varepsilon}{2}\|\Delta u\|_2^2$. Hence

$$\begin{aligned}
e^*_{\overline{u},\overline{v}}(u,v)&\ge(1-2\varepsilon)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{\varepsilon}{2}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)\\
&\quad+M\left(\|\nabla\overline{u}\|_2^2\right)\|\nabla u\|_2^2+M\left(\|\nabla\overline{v}\|_2^2\right)\|\nabla v\|_2^2.
\end{aligned}$$

Choosing $\varepsilon = \frac{2}{5}$, we have

$$e^*_{\overline{u},\overline{v}}(u,v) \geq \frac{1}{5}e(u,v). \tag{3.25}$$

From (3.23) and (3.25), we obtain

$$\frac{d}{dt}e^*_{\overline{u},\overline{v}}(u(t),v(t)) \leq 10c_0LR^2e^*_{\overline{u},\overline{v}}(u(t),v(t)) + \frac{14\sqrt{5}}{5}c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e^*_{\overline{u},\overline{v}}(u(t),v(t))^{\frac{1}{2}}.$$

By Gronwall Lemma, we get

$$e^*_{\overline{u},\overline{v}}(u(t),v(t)) \leq \left(e^*_{\overline{u}(0),\overline{v}(0)}(u_0,v_0)^{\frac{1}{2}} + \frac{7\sqrt{5}}{5}c_1\left(R^{\alpha+1}+R^{\beta+1}\right)T\right)^2 \mathrm{e}^{5c_0LR^2T}. \tag{3.26}$$

Note that from (3.24), and Young′s inequality, we have

$$e^*_{\overline{u}(0),\overline{v}(0)}(u(0),v(0)) \leq c_2, \tag{3.27}$$

where

$$\begin{aligned} c_2 = 2&\left(\|u_1\|_2^2 + \|v_1\|_2^2\right) + \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 \\ &+M\left(\|\nabla u_0\|_2^2\right)\|\nabla u_0\|_2^2 + M\left(\|\nabla v_0\|_2^2\right)\|\nabla v_0\|_2^2. \end{aligned}$$

Then, from (3.25), (3.26), (3.27), we obtain for any $t \in (0,T]$,

$$\begin{aligned} e(u(t),v(t)) &\leq 5e^*_{\overline{u},\overline{v}}(u(t),v(t)) \\ &\leq \chi(u_0,u_1,v_0,v_1,R,T)^2\mathrm{e}^{5c_0LR^2T}, \end{aligned} \tag{3.28}$$

where

$$\chi(u_0,u_1,v_0,v_1,R,T) = c_2^{\frac{1}{2}} + \frac{7\sqrt{5}}{5}c_1\left(R^{\alpha+1}+R^{\beta+1}\right)T.$$

We see that for parameters T and R satisfy

$$\chi(u_0,u_1,v_0,v_1,R,T)^2\mathrm{e}^{5c_0LR^2T} \leq R^2. \tag{3.29}$$

Then S maps $X_{T,R}$ into itself. By Theorem 3.1, $w \in C^0([0,T];H^2(\Omega)\cap H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. Moreover, it follows from (3.19) and (3.29) that $u', v' \in L^2((0,T);H_0^1(\Omega))$. Hence, we see that $u(t)$ and $v(t)$ belong to $C^0([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$.

Next, we will show that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. Let $(\overline{u_i},\overline{v_i}) \in X_{T,R}$ and (u_i,v_i) be the corresponding solution to (3.3)−(3.8). By the above discussion, we see that $(u_i,v_i) \in X_{T,R}$, $i = 1,2$. Setting $w_1(t) = (u_1-u_2)(t)$, $w_2(t) = (v_1-v_2)(t)$, then w_1 and w_2 satisfy the following system:

$$\begin{aligned} &(w_1)_{tt} - M\left(\|\nabla\overline{u_1}\|_2^2\right)\Delta w_1 - \Delta(w_1)_t \\ &\quad = f_1(\overline{u_1},\overline{v_1}) - f_1(\overline{u_2},\overline{v_2}) + \left[M\left(\|\nabla\overline{u_1}\|_2^2\right) - M\left(\|\nabla\overline{u_2}\|_2^2\right)\right]\Delta u_2, \end{aligned} \tag{3.30}$$

$$\begin{aligned}&(w_2)_{tt} - M\left(\|\nabla\overline{v_1}\|_2^2\right)\Delta w_2 - \Delta\left(w_2\right)_t \\ &\quad = f_2(\overline{u_1},\overline{v_1}) - f_2(\overline{u_2},\overline{v_2}) + \left[M\left(\|\nabla\overline{v_1}\|_2^2\right) - M\left(\|\nabla\overline{v_2}\|_2^2\right)\right]\Delta v_2,\end{aligned} \tag{3.31}$$

$$w_1(0) = 0, \;\; \left(w_1\right)_t(0) = 0, \tag{3.32}$$

$$w_2(0) = 0, \;\; \left(w_2\right)_t(0) = 0. \tag{3.33}$$

Multiplying (3.30) by $2\left(w_1\right)_t$, and integrating it over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\left\{\|(w_1)_t\|_2^2 + M\left(\|\nabla\overline{u_1}\|_2^2\right)\|\nabla w_1\|_2^2\right\} + 2\left\|\nabla\left(w_1\right)_t\right\|_2^2 \\ &\quad = I_{u3} + I_{u4} + I_{u5},\end{aligned} \tag{3.34}$$

where

$$I_{u3} = \left(\frac{d}{dt}M\left(\|\nabla\overline{u_1}\|_2^2\right)\right)\|\nabla w_1\|_2^2, \tag{3.35}$$

$$I_{u4} = 2\left[M\left(\|\nabla\overline{u_1}\|_2^2\right) - M\left(\|\nabla\overline{u_2}\|_2^2\right)\right]\int_\Omega \Delta u_2\left(w_1\right)_t dx, \tag{3.36}$$

$$I_{u5} = 2\int_\Omega \left(f_1(\overline{u_1},\overline{v_1}) - f_1(\overline{u_2},\overline{v_2})\right)\left(w_1\right)_t dx. \tag{3.37}$$

Similarly, we have

$$\begin{aligned}&\frac{d}{dt}\left\{\|(w_2)_t\|_2^2 + M\left(\|\nabla\overline{v_1}\|_2^2\right)\|\nabla w_2\|_2^2\right\} + 2\left\|\nabla\left(w_2\right)_t\right\|_2^2 \\ &\quad = I_{v3} + I_{v4} + I_{v5},\end{aligned} \tag{3.38}$$

where

$$I_{v3} = \left(\frac{d}{dt}M\left(\|\nabla\overline{v_1}\|_2^2\right)\right)\|\nabla w_2\|_2^2, \tag{3.39}$$

$$I_{v4} = 2\left[M\left(\|\nabla\overline{v_1}\|_2^2\right) - M\left(\|\nabla\overline{v_2}\|_2^2\right)\right]\int_\Omega \Delta v_2\left(w_2\right)_t dx, \tag{3.40}$$

$$I_{v5} = 2\int_\Omega \left(f_2(\overline{u_1},\overline{v_1}) - f_2(\overline{u_2},\overline{v_2})\right)\left(w_2\right)_t dx. \tag{3.41}$$

From (3.35), by using the Divergence theorem and Lemma 2.1, we have

$$\begin{aligned}|I_{u3}| &\le 2L\|\Delta\overline{u_1}\|_2\,\|(\overline{u_1})_t\|_2\,\|\nabla w_1\|_2^2 \\ &\le c_0LR^2e(w_1,w_2).\end{aligned} \tag{3.42}$$

Note that by Lemma 2.1, we have

$$\begin{aligned}\left|M\left(\|\nabla\overline{u_1}\|_2^2\right) - M\left(\|\nabla\overline{u_2}\|_2^2\right)\right| &\le L\left(\|\nabla\overline{u_1}\|_2 + \|\nabla\overline{u_2}\|_2\right)\|\nabla\overline{u_1} - \nabla\overline{u_2}\|_2 \\ &\le 2c_*^2RLe(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}.\end{aligned} \tag{3.43}$$

From (3.36) and (3.43), we get

$$|I_{u4}| \leq 4c_*^2 LR^2 e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}} e(w_1,w_2)^{\frac{1}{2}}. \tag{3.44}$$

By (A2) and Lemma 2.1, we get

$$|I_{u5}| \leq 4c_1\left(R^{\alpha}+R^{\beta}\right) e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}} e(w_1,w_2)^{\frac{1}{2}}. \tag{3.45}$$

Hence from $(3.34),(3.42),(3.44),(3.45)$, we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\|(w_1)_t\|_2^2 + M\left(\|\nabla\overline{u_1}\|_2^2\right)\|\nabla w_1\|_2^2\right\} + 2\|\nabla(w_1)_t\|_2^2 \\
&\quad \leq c_0LR^2e(w_1,w_2) + 4c_*^2LR^2e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}} \\
&\qquad +4c_1\left(R^{\alpha}+R^{\beta}\right)e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}.
\end{aligned} \tag{3.46}$$

By the same procedure, from (3.38), we have the similar inequality for w_2. Hence, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\|(w_1)_t\|_2^2 + M\left(\|\nabla\overline{u_1}\|_2^2\right)\|\nabla w_1\|_2^2 + \|(w_2)_t\|_2^2 + M\left(\|\nabla\overline{v_1}\|_2^2\right)\|\nabla w_2\|_2^2\right\} \\
&\qquad +2\|\nabla(w_1)_t\|_2^2 + 2\|\nabla(w_2)_t\|_2^2 \\
&\quad \leq 2c_0LR^2e(w_1,w_2) + 8c_*^2LR^2e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}} \\
&\qquad +8c_1\left(R^{\alpha}+R^{\beta}\right)e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}.
\end{aligned} \tag{3.47}$$

On the other hand, multiplying (3.30) by $-2\Delta w_{1,}$ and integrating it over Ω, and then by Divergence theorem and (A2), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta w_1\|_2^2 - 2\int_{\Omega}(w_1)_t\,\Delta w_1 dx\right\} + 2M\left(\|\nabla\overline{u_1}\|_2^2\right)\|\Delta w_1\|_2^2 \\
&\quad \leq 2\|\nabla(w_1)_t\|_2^2 + 4c_*^2LR^2e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}} \\
&\qquad +2c_1\left(R^{\alpha}+R^{\beta}\right)e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}.
\end{aligned}$$

The similar inequality is obtained for w_2. Therefore, we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta w_1\|_2^2 - 2\int_{\Omega}(w_1)_t\,\Delta w_1 dx + \|\Delta w_2\|_2^2 - 2\int_{\Omega}(w_2)_t\,\Delta w_2 dx\right\} \\
&\qquad +2M\left(\|\nabla\overline{u_1}\|_2^2\right)\|\Delta w_1\|_2^2 + 2M\left(\|\nabla\overline{v_1}\|_2^2\right)\|\Delta w_2\|_2^2 \\
&\quad \leq 2\left[\|\nabla(w_1)_t\|_2^2 + \|\nabla(w_2)_t\|_2^2\right] + 8c_*^2LR^2e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}} \\
&\qquad +4c_1\left(R^{\alpha}+R^{\beta}\right)e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}.
\end{aligned} \tag{3.48}$$

Multiplying (3.48) by ε, $0<\varepsilon\leq 1$, and adding it to (3.44), we have

$$\begin{aligned}
&\frac{d}{dt}e^*_{\overline{u_1},\overline{v_1}}(w_1,w_2) + 2(1-\varepsilon)\left[\|\nabla(w_1)_t\|_2^2 + \|\nabla(w_2)_t\|_2^2\right] \\
&\quad \leq 2c_0LR^2e(w_1,w_2) + 8c_*^2(1+\varepsilon)LR^2e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}} + \\
&\qquad 4(1+\varepsilon)c_1\left(R^{\alpha+1}+R^{\beta+1}\right)e(\overline{u_1}-\overline{u_2},\overline{v_1}-\overline{v_2})^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}},
\end{aligned} \tag{3.49}$$

where $e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2)$ is given by (3.24) with $u = w_1$, $v = w_2$, $\overline{u} = \overline{u_1}$ and $\overline{v} = \overline{v_1}$. Taking $\varepsilon = \frac{2}{5}$ in (3.49), and as in (3.22)−(3.25), we have

$$e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2) \geq \frac{1}{5} e(w_1, w_2). \tag{3.50}$$

Therefore, from (3.49) and (3.50), we get

$$\begin{aligned}\frac{d}{dt} e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2) &\leq 10c_0 LR^2 e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2) + c_3 LR^2 e(\overline{u_1} - \overline{u_2}, \overline{v_1} - \overline{v_2})^{\frac{1}{2}} e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2)^{\frac{1}{2}} \\ &\quad + c_4 \left(R^{\alpha+1} + R^{\beta+1}\right) e(\overline{u_1} - \overline{u_2}, \overline{v_1} - \overline{v_2})^{\frac{1}{2}} e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2)^{\frac{1}{2}},\end{aligned}$$

where $c_3 = \frac{56\sqrt{5}}{5} c_*^2$, $c_4 = \frac{28\sqrt{5}}{5} c_1$. Noting that $e^*_{\overline{u_1}(0),\overline{v_1}(0)}(w_1(0), w_2(0)) = 0$, and by applying Gronwall Lemma, we get

$$e^*_{\overline{u_1},\overline{v_1}}(w_1, w_2) \leq \left[\frac{c_3}{2} LR^2 + \frac{c_4}{2}\left(R^{\alpha+1} + R^{\beta+1}\right)\right]^2 T^2 \mathrm{e}^{5c_0 LR^2 T} \sup_{0 \leq t \leq T} e(\overline{u_1} - \overline{u_2}, \overline{v_1} - \overline{v_2}).$$

By (3.2) and (3.50), we have

$$d\left((u_1, v_1), (u_2, v_2)\right) \leq C(T, R)^{\frac{1}{2}} d\left((\overline{u_1}, \overline{v_1}), (\overline{u_2}, \overline{v_2})\right),$$

where

$$C(T, R) = 5\left[\frac{c_3}{2} LR^2 + \frac{c_3}{2}\left(R^{\alpha+1} + R^{\beta+1}\right)\right] T \mathrm{e}^{5c_0 LR^2 T}. \tag{3.51}$$

Hence, under inequality (3.29), S is a contraction mappingn if $C(T, R) < 1$. Indeed, we choose R sufficient large and T sufficient small so that (3.29) and (3.51) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

4. Blow-up property

In this section, we will study blow-up phenomena of solutions for a system (1.1)−(1.6). In order to state our results, we further make the following assumptions:

(A4) there exists a positive constant $\delta > 0$ such that

$$uf_1(u, v) + vf_2(u, v) \geq (2 + 4\delta) F(u, v), \text{ for all } u, v \in R,$$

where $F(u, v)$ is given in (A1).

(A5) $(2\delta + 1)\overline{M}(s) \geq M(s)s$, for all $s \geq 0$, and δ is the constant given in (A4), where $\overline{M}(s) = \int_0^s M(r)\, dr$.

Definition. A solution $w(t) = (u(t), v(t))$ of (1.1)−(1.6) is called blow-up if there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx = \infty.$$

Let $(u(t), v(t))$ be the solution of (1.1)$-$(1.6), define the energy function

$$E(t) = \frac{1}{2}\left[\|u_t\|_2^2 + \|v_t\|_2^2 + \overline{M}\left(\|\nabla u\|_2^2\right) + \overline{M}\left(\|\nabla v\|_2^2\right)\right] - \int_\Omega F(u,v)dx,\ t \geq 0. \tag{4.1}$$

Lemma 4.1. *Assume that* (A1) *and* (A3) *hold, then* $E(t)$ *is a nonincreasing function and*

$$E\left(t\right) = E\left(0\right) - \int_0^t \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) dt. \tag{4.2}$$

Proof. By differentiating (4.1) and using (1.1), (1.2), (A1) and (A3), we get

$$\frac{dE(t)}{dt} = -\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right).$$

Thus, Lemma 4.1 follows at once.

Now, let

$$a\left(t\right) = \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) dt,\ t \geq 0. \tag{4.3}$$

Lemma 4.2. *Assume that* (A1), (A3), (A4) *and* (A5) *hold, we have*

$$\begin{aligned} &a''\left(t\right) - 4\left(\delta+1\right)\int_\Omega \left(u_t^2 + v_t^2\right) dx \\ &\quad \geq \left(-4-8\delta\right)E\left(0\right) + \left(4+8\delta\right)\int_0^t \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) dt. \end{aligned} \tag{4.4}$$

Proof. Form (4.3), we have

$$a'\left(t\right) = 2\int_\Omega \left(uu_t + vv_t\right) dx + \|\nabla u\|_2^2 + \|\nabla v\|_2^2. \tag{4.5}$$

By $(1.1), (1.2)$ and Divergence theorem, we get

$$\begin{aligned} a''\left(t\right) = {}& 2\int_\Omega \left(u_t^2 + v_t^2\right) dx - 2\left(M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + M\left(\|\nabla v\|_2^2\right)\|\nabla v\|_2^2\right) \\ &+2\int_\Omega \left(uf_1(u,v) + vf_2(u,v)\right) dx. \end{aligned} \tag{4.6}$$

By (4.2), we have from (4.6)

$$
\begin{aligned}
& a''(t) - 4(\delta+1)\int_\Omega \left(u_t^2+v_t^2\right)dx \\
&= (-4-8\delta)E(0) + (4+8\delta)\int_0^t \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right)ds \\
&\quad + \left[(2+4\delta)\overline{M}\left(\|\nabla u\|_2^2\right) - 2M\left(\|\nabla u\|_2^2\right)\int_\Omega |\nabla u|^2\,dx\right] \\
&\quad + \left[(2+4\delta)\overline{M}\left(\|\nabla v\|_2^2\right) - 2M\left(\|\nabla v\|_2^2\right)\int_\Omega |\nabla v|^2\,dx\right] \\
&\quad + \int_\Omega 2\left[uf_1(u,v) + vf_2(u,v) - (2+4\delta)F(u,v)\right]dx.
\end{aligned}
$$

Therefore, from (A4) and (A5), we obtain (4.4). Now, we consider three different cases on the sign of the initial energy $E(0)$. (1) If $E(0)<0$, then from (4.4), we have

$$a'(t) \geq a'(0) - 4(1+2\delta)E(0)\,t,\ \ t\geq 0.$$

Thus we get $a'(t) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2$ for $t>t^*$, where

$$t^* = \max\left\{\frac{a'(0) - \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)}{4(1+2\delta)E(0)},\ 0\right\}. \tag{4.7}$$

(2) If $E(0)=0$, then $a''(t)\geq 0$ for $t\geq 0$. If $a'(0) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2$, then we have $a'(t) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2,\ t\geq 0$. (3) For the case that $E(0)>0$, we first note that

$$2\int_0^t\int_\Omega \nabla u \nabla u_t\,dxdt = \|\nabla u\|_2^2 - \|\nabla u_0\|_2^2. \tag{4.8}$$

By Hölder inequality and Young′s inequality, we have from (4.8)

$$\|\nabla u\|_2^2 \leq \|\nabla u_0\|_2^2 + \int_0^t \|\nabla u\|_2^2 dt + \int_0^t \|\nabla u_t\|_2^2 dt. \tag{4.9}$$

In the same way, we get

$$\|\nabla v\|_2^2 \leq \|\nabla v_0\|_2^2 + \int_0^t \|\nabla v\|_2^2 dt + \int_0^t \|\nabla v_t\|_2^2 dt. \tag{4.10}$$

By Hölder inequality, Young′s inequality and then using (4.9) and (4.10), we have from (4.5)

$$a'(t) \leq a(t) + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + \int_\Omega \left(u_t^2+v_t^2\right)dx + \int_0^t \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right)dt. \tag{4.11}$$

Hence by (4.4) and (4.11), we obtain

$$a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \geq 0,$$

where

$$K_1 = (4+8\delta)E(0) + 4(\delta+1)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) .$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)},\ t>0.$$

Then $b(t)$ satisfies (2.2). By Lemma 2.2, we see that if

$$a'(0) > r_2\left[a(0) + \frac{K_1}{4(1+\delta)}\right] + \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right), \tag{4.12}$$

then $a'(t) > \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)$, $t>0$, where r_2 is given in Lemma 2.2. Consequently, we have

Lemma 4.3. *Assume that* (A1), (A3), (A4) *and* (A5) *hold and that either one of the following statements is satisfied :*
(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)$,
(iii) $E(0) > 0$ *and* (4.12) *holds,*
then $a'(t) > \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)$ for $t > t_0$, *where* $t_0 = t^*$ *is given by* (4.7) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Now, we will find the estimate for the life span of $a(t)$. Let

$$J(t) = \left[a(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]^{-\delta},\ \text{for } t \in [0, T_1], \tag{4.13}$$

where $T_1 > 0$ is a certain constant which will be specified later. Then we have

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}}\left(a'(t) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right),$$

and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \tag{4.14}$$

where

$$\begin{aligned} V(t) = {} & a''(t)\left[a(t) + (T_1 - t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right] \\ & - (1+\delta)\left(a'(t) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2 . \end{aligned} \tag{4.15}$$

For simplicity of calculation, we denote

$$
\begin{aligned}
P_u &= \int_\Omega u^2 dx,\ P_v = \int_\Omega v^2 dx,\\
Q_u &= \int_0^t \|\nabla u\|_2^2\, dt,\ Q_v = \int_0^t \|\nabla v\|_2^2\, dt,\\
R_u &= \int_\Omega u_t^2 dx,\ R_v = \int_\Omega v_t^2 dx,\\
S_u &= \int_0^t \|\nabla u_t\|_2^2\, dt,\ S_v = \int_0^t \|\nabla v_t\|_2^2\, dt.
\end{aligned}
$$

From (4.5), (4.8), and Hölder inequality, we get

$$
\begin{aligned}
a'(t) &= 2\int_\Omega (uu_t + vv_t)\, dx + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\\
&\quad +2\int_0^t\int_\Omega \nabla u \nabla u_t dx dt + 2\int_0^t\int_\Omega \nabla v \nabla v_t dx dt\\
&\le 2(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{R_vP_v} + \sqrt{Q_vS_v})\\
&\quad + \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2.
\end{aligned} \tag{4.16}
$$

By (4.4), we have

$$
a''(t) \ge (-4-8\delta)\, E(0) + 4(1+\delta)(R_u + S_u + R_v + S_v). \tag{4.17}
$$

Thus, from (4.16), (4.17), (4.15) and (4.13), we obtain

$$
\begin{aligned}
V(t) &\ge [(-4-8\delta)\, E(0) + 4(1+\delta)(R_u + S_u + R_v + S_v)]\, J(t)^{-\frac{1}{\delta}}\\
&\quad -4(1+\delta)\left(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{R_vP_v} + \sqrt{Q_vS_v}\right)^2.
\end{aligned}
$$

And by (4.13) and (4.3), we have

$$
\begin{aligned}
V(t) &\ge (-4-8\delta)\, E(0)\, J(t)^{-\frac{1}{\delta}}\\
&\quad +4(1+\delta)\left[(R_u + S_u + R_v + S_v)(T_1 - t)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) + \Theta(t)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta(t) &= (R_u + S_u + R_v + S_v)(P_u + Q_u + P_v + Q_v)\\
&\quad -\left(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{R_vP_v} + \sqrt{Q_vS_v}\right)^2.
\end{aligned}
$$

By Schwarz inequality, $\Theta(t)$ is nonnegative. Hence, we have

$$
V(t) \ge (-4-8\delta)\, E(0)\, J(t)^{-\frac{1}{\delta}},\ t \ge t_0. \tag{4.18}
$$

Therefore by (4.14) and (4.18), we get

$$J''(t) \le \delta(4+8\delta)E(0)J(t)^{1+\frac{1}{\delta}}, \ t \ge t_0. \tag{4.19}$$

Note that by Lemma 4.3, $J'(t)<0$ for $t>t_0$. Mutiplying (4.19) by $J'(t)$ and integrating it from t_0 to t, we get

$$J'(t)^2 \ge \alpha + \beta J(t)^{2+\frac{1}{\delta}} \text{ for } t \ge t_0,$$

where

$$\alpha = \delta^2 J(t_0)^{2+\frac{2}{\delta}}\left[(a'(t_0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2)^2 - 8E(0)J(t_0)^{\frac{-1}{\delta}}\right], \tag{4.20}$$

and

$$\beta = 8\delta^2 E(0). \tag{4.21}$$

We observe that

$$\alpha > 0 \text{ iff } E(0) < \frac{\left(a'(t_0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]}.$$

Then by Lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} J(t) = 0$ and the upper bound of T^* is estimated respectively according to the sign of $E(0)$. This means that

$$\lim_{t\to T^{*-}}\left\{\int_\Omega \left(u^2+v^2\right)dx + \int_0^t \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)dt\right\} = \infty.$$

By Poincaré inequality, it implies that

$$\lim_{t\to T^{*-}} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right)dx = \infty. \tag{4.22}$$

Theorem 4.4. *Assume that* (A1), (A3), (A4) *and* (A5) *hold and that either one of the following statements is satisfied :*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)$,

(iii) $0 < E(0) < \frac{\left(a'(t_0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]}$ *and* (4.12) *holds,*

then the solution $(u(t), v(t))$ *blows up at finite time* T^* *in the sense of* (4.22).

In case (i),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{4.23}$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\alpha}{-\beta}}\right\}$, we have

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}. \tag{4.24}$$

In case (ii),

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)} \tag{4.25}$$

or

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{4.26}$$

In case (iii),

$$T^* \le \frac{J(t_0)}{\sqrt{\alpha}} \tag{4.27}$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}, \tag{4.28}$$

where $c = \left(\frac{\alpha}{\beta}\right)^{2+\frac{1}{\delta}}$, here α and β are given in (4.20), (4.21). Note that in case (i), $t_0 = t^*$ is given in (4.7) and $t_0 = 0$ in case (ii) and (iii). The choice of T_1 in (4.13) is possible under some conditions. We shall discuss it in the following Remark.

Remark 4.5.

(i) For the case $E(0) = 0$

(1) If $2\delta \int_\Omega (u_0 u_1 + v_0 v_1)\, dx - \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) > 0$, by (4.25), we choose

$$T_1 \ge -\frac{J(0)}{J'(0)}.$$

Then, in particular, we have

$$T^* \le T_1 = \omega,$$

where

$$\omega = \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{2\delta \int_\Omega (u_0 u_1 + v_0 v_1)\, dx - (\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2)}.$$

(2) If $0 < 2\delta \int_\Omega (u_0 u_1 + v_0 v_1)\, dx \le \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)$, by (4.26), we choose

$$T_1 \ge \frac{J(0)}{\sqrt{\alpha}}. \tag{4.29}$$

From (4.29), and by Young′s inequality, we get

$$\begin{aligned} &\|u_0\|_2^2 + \|v_0\|_2^2 + T_1 \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) \\ &\quad \le \delta T_1 \left(\|u_0\|_2^2 + \|u_1\|_2^2 + \|v_0\|_2^2 + \|v_1\|_2^2\right), \\ &\quad \le \overline{c} T_1 \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + \|u_1\|_2^2 + \|v_1\|_2^2\right), \end{aligned}$$

where the last inequality is obtained by Lemma 2.1, with $\overline{c} = \max\left(\delta c_*^2,\ 1\right)$. Then, in particular, we have

$$T^* \leq T_1 = \rho,$$

where

$$\rho = \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{(\overline{c}-1)\left[\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right] + \overline{c}\left(\|u_1\|_2^2 + \|v_1\|_2^2\right)}.$$

(ii) For the case $E(0) < 0$

(1) If $\int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0$, then $a'(t) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2$ and $t^* = 0$ (by (4.7)). Thus T_1 can be chosen as in (i).

(2) If $\int_\Omega (u_0 u_1 + v_0 v_1)\, dx \leq 0$, then $t^* = \frac{a'(0) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2}{4(1+2\delta)E(0)}$. Thus, by (4.23), we choose $T_1 \geq t^* - \frac{J(t^*)}{J'(t^*)}$.

(iii) For the case $E(0) > 0$By (4.27), we choose

$$T_1 = \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{\left[c\delta r_2^2\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) - 1\right]\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)},$$

here

$$c = 1, \quad \text{if } \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 > \frac{1}{\delta r_2^2},$$

$$c = \frac{2}{\delta r_2^2\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)}, \quad \text{if } \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \leq \frac{1}{\delta r_2^2}.$$

Under the condition

$$E(0) < \min\{\kappa_1, \kappa_2\},$$

where

$$\kappa_1 = \frac{(1+\delta)\left[a'(0) - r_2 a(0) - (r_2+1)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\right]}{r_2(1+2\delta)},$$

and

$$\kappa_2 = \frac{\left(\int_\Omega (u_0 u_1 + v_0 v_1) dx\right)^2 \left[c\delta r_2^2\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) - 1\right]}{2c\delta r_2^2\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)}.$$

Then, we have

$$T^* \leq \frac{\mu}{\sqrt{4(\int_\Omega u_0 u_1 + v_0 v_1 dx)^2 - 8E(0)\mu}},$$

where

$$\mu = \frac{c\delta r_2^2\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)}{c\delta r_2^2\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right) - 1}.$$

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