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NEW RELATIONS AMONG EULER SUMS OF EVEN WEIGHT

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Abstract. In this paper, we shall consider different kinds of Euler sums which are related to Ramanujan's constant $G(1)$. We develop new relations among these Euler sums and classical Euler sums of even weight. In particular, from these relations we give explicit evaluations of

$$G_{1,2n+1} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \sum_{j=1}^{k} \frac{1}{j}.$$

1. Introduction

For a pair of positive integers p and q with $q > 1$, the classical Euler sum is defined as

$$S_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}. \tag{1.1}$$

See for example [1, 2, 3, 5] for the details. The following well-known formula for $S_{1,n}$ is due to Euler (cf. [1, p. 253]).

Theorem A. *For each positive integer n with $n \geq 2$, we have*

$$S_{1,n} = \frac{n+2}{2}\zeta(n+1) - \frac{1}{2}\sum_{j=2}^{n-1} \zeta(j)\zeta(n+1-j). \tag{1.2}$$

In an attempt to investigate relations among Euler sums of the same even weight, Chen and Eie [4] defined, for positive integers p and q with $q \geq 2$, some related new sums as

$$G_{p,q} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^q} \sum_{j=1}^{k} \frac{1}{j^p}, \tag{1.3}$$

$$H_{p,q} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^q} \sum_{j=0}^{k} \frac{1}{(2j+1)^p}, \tag{1.4}$$

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and for positive integers r,

$$E_{p,q}^{(r)} = \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{kr} \frac{1}{j^p}. \tag{1.5}$$

These generalized Euler sums come naturally from $S_{p,q}$. For example,

$$\begin{aligned} S_{p,q} &= \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p} \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k)^q} \sum_{j=1}^{2k} \frac{1}{j^p} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)^q} \sum_{j=1}^{2k+1} \frac{1}{j^p} \\ &= 2^{-q} E_{p,q}^{(2)} + H_{p,q} + 2^{-p} G_{p,q}. \end{aligned} \tag{1.6}$$

In his famous notebooks [1, p. 252], Ramanujan claimed that

$$G(1) := \sum_{k=1}^{\infty} \frac{1}{8k^3} \sum_{j=1}^{k} \frac{1}{2j-1} = \frac{\pi}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)^3} - \frac{\pi}{3\sqrt{3}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}. \tag{1.7}$$

The above formula is incorrect as can be easily seen numerically. In our notation we have

$$8G(1) = E_{1,3}^{(2)} - \frac{1}{2} S_{1,3} = E_{1,3}^{(2)} - \frac{5}{8}\zeta(4) = G_{2,2} + \frac{15}{16}\zeta(4).$$

There is no closed-form evaluation known for $E_{1,2n+1}^{(2)}$, nor for $G_{p,q}$ if $p+q$ is even. However, we are able to evaluate $G_{1,2n+1}$ and $E_{1,2n+1}^{(2)} + 2^{2n+1} H_{1,2n+1}$. Here are our main theorems in this paper.

Theorem 1. *For each positive integer n, we have*

$$\begin{aligned} G_{1,2n+1} = {} & (2 - 2^{-(2n+1)}) \{S_{1,2n+1} - \zeta(2+2n)\} \\ & - 2\lambda(2n+1)\log 2 + \sum_{\ell=2}^{2n} 2^{1-\ell} \zeta(\ell) \lambda(2n+2-\ell), \end{aligned} \tag{1.8}$$

where

$$\lambda(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s} = (1 - 2^{-s})\zeta(s). \tag{1.9}$$

Theorem 2. *For each positive integer n, we have*

$$\begin{aligned} & E_{1,2n+1}^{(2)} + 2^{2n+1} H_{1,2n+1} \\ & \quad = \frac{1}{2} S_{1,2n+1} + \left(2^{2n+1} - \frac{1}{2}\right) \zeta(2n+2) + 2^{2n+1} \lambda(2n+1) \log 2 \\ & \quad\quad - \sum_{\ell=2}^{2n} 2^{2n+1-\ell} \zeta(\ell) \lambda(2n+2-\ell). \end{aligned} \tag{1.10}$$

As a byproduct of our results, we also prove the following relations among Euler sums of the same even weight.

Theorem 3. *For each positive integer* n,

$$\sum_{\ell=2}^{2n} 2^{\ell-1} S_{\ell,2n+2-\ell} = (2^{2n}-1)S_{1,2n+1} - 2^{2n}\zeta(2n+2) + \sum_{\ell=2}^{2n} 2^{2n+1-\ell}\zeta(\ell)\lambda(2n+2-\ell). \tag{1.11}$$

Two more general theorems will be given in Section 5.

2. The evaluation of $G_{1,2n+1}$

As shown in (1.6), we already have a formula relating $G_{1,2n+1}$ to $E^{(2)}_{1,2n+1} + 2^{2n+1} H_{1,2n+1}$. We need one more relation in order to evaluate $G_{1,2n+1}$. First we obtain an evaluation of $G_{1,2n+1}$ in terms of $S_{\ell,2n+2-\ell}$ for $\ell = 2, 3, \ldots, 2n$.

Proposition 1. *For each positive integer* n, *we have*

$$G_{1,2n+1} = (1+2^{-(2n+1)})S_{1,2n+1} - (1-2^{-(2n+1)})\zeta(2n+2) - 2\lambda(2n+1)\log 2 + 2^{-2n}\sum_{\ell=2}^{2n} 2^{\ell-1} S_{\ell,2n+2-\ell}. \tag{2.1}$$

Proof. We begin with $E^{(2)}_{1,2n+1}$ defined by

$$E^{(2)}_{1,2n+1} = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=1}^{2k} \frac{1}{j}. \tag{2.2}$$

Note that

$$\begin{aligned} E^{(2)}_{1,2n+1} - S_{1,2n+1} &= \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=k+1}^{2k} \frac{1}{j} \\ &= \frac{1}{2}\zeta(2n+2) + \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=1}^{k-1} \frac{1}{j+k} \\ &= \frac{1}{2}\zeta(2n+2) + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(k+j)^{2n+1}(k+2j)}. \end{aligned}$$

In light of the partial fraction decomposition

$$\frac{1}{(x+k)^{2n+1}(2x+k)} = \frac{2^{2n}}{k^{2n+1}}\left\{\frac{1}{x+k/2}-\frac{1}{x+k}\right\} - \frac{1}{k(x+k)^{2n+1}}$$
$$-\sum_{\ell=2}^{2n}\frac{2^{\ell-1}}{k^{\ell}(x+k)^{2n+2-\ell}},$$

we conclude that

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(k+j)^{2n+1}(k+2j)}$$
$$= 2^{2n}\sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\sum_{j=1}^{\infty}\left(\frac{1}{j+k/2}-\frac{1}{j+k}\right) - \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k(k+j)^{2n+1}}$$
$$-\sum_{\ell=2}^{2n}2^{\ell-1}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k^{\ell}(k+j)^{2n+2-\ell}}. \tag{2.3}$$

The second term in the above is equal to

$$-\left\{S_{1,2n+1}-\zeta(2n+2)\right\},$$

while the third term is equal to

$$-\sum_{\ell=2}^{2n}2^{\ell-1}\left\{S_{\ell,2n+2-\ell}-\zeta(2n+2)\right\},$$

or

$$-\sum_{\ell=2}^{2n}2^{\ell-1}S_{\ell,2n+2-\ell}+(2^{2n}-2)\zeta(2n+2).$$

Now it remains to evaluate the first term. When k ranges over all positive even integers, the corresponding partial sum is equal to

$$2^{2n}\sum_{k=1}^{\infty}\frac{1}{(2k)^{2n+1}}\sum_{j=1}^{\infty}\left\{\frac{1}{j+k}-\frac{1}{j+2k}\right\},$$

which is equal to

$$\frac{1}{2}\left\{E^{(2)}_{1,2n+1}-S_{1,2n+1}\right\}.$$

When k ranges over all positive odd integers, the corresponding partial sum is

$$2^{2n}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+1}}\sum_{j=1}^{\infty}\left\{\frac{1}{j+k+1/2}-\frac{1}{j+2k+1}\right\}. \tag{2.4}$$

In view of the Kronecker limit formula for Hurwitz zeta function [1],

$$\lim_{s\to 1^+}\left\{\zeta(s,\delta)-\frac{1}{s-1}\right\}=-\frac{\Gamma'(\delta)}{\Gamma(\delta)}, \tag{2.5}$$

we get in particular that

$$\sum_{j=1}^{\infty}\left\{\frac{1}{j+k+1/2}-\frac{1}{j+2k+1}\right\}=-\frac{\Gamma'(k+3/2)}{\Gamma(k+3/2)}+\frac{\Gamma'(2k+2)}{\Gamma(2k+2)}. \tag{2.6}$$

Also we have for positive integers k (e.g. see [1]),

$$-\frac{\Gamma'(k+3/2)}{\Gamma(k+3/2)}=-2\left(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\right)+\gamma+2\log 2 \tag{2.7}$$

and

$$\frac{\Gamma'(2k+2)}{\Gamma(2k+2)}=1+\frac{1}{2}+\cdots+\frac{1}{2k+1}-\gamma, \tag{2.8}$$

where γ is the Euler constant defined by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right). \tag{2.9}$$

Consequently, this expression in (2.4) is equal to

$$2^{2n+1}\lambda(2n+1)\log 2-2^{2n}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+1}}\sum_{j=1}^{2k+1}\frac{(-1)^{j+1}}{j},$$

which we rewrite as

$$2^{2n+1}\lambda(2n+1)\log 2-2^{2n}H_{1,2n+1}+2^{2n-1}G_{1,2n+1}.$$

It follows that

$$\begin{aligned}\frac{1}{2}E^{(2)}_{1,2n+1}=&-\frac{1}{2}S_{1,2n+1}+\left(2^{2n}-\frac{1}{2}\right)\zeta(2n+2)+2^{2n+1}\lambda(2n+1)\log 2\\&-2^{2n}H_{1,2n+1}+2^{2n-1}G_{1,2n+1}-\sum_{\ell=2}^{2n}2^{\ell-1}S_{\ell,2n+2-\ell}.\end{aligned}$$

Our assertion then follows from the further relation

$$\frac{1}{2}\left\{E^{(2)}_{1,2n+1}+2^{2n+1}H_{1,2n+1}\right\}=2^{2n}S_{1,2n+1}-2^{2n-1}G_{1,2n+1}. \tag{2.10}$$

As an immediate consequence, we have

Proposition 2. *For each positive integer n,*

$$E^{(2)}_{1,2n+1} + 2^{2n+1} H_{1,2n+1} = \left(2^{2n} - \frac{1}{2}\right) S_{1,2n+1} + \left(2^{2n} - \frac{1}{2}\right) \zeta(2n+2) + 2^{2n+1}\lambda(2n+1)\log 2 - \sum_{\ell=2}^{2n} 2^{\ell-1} S_{\ell,2n+2-\ell}.$$

Rremark. Unfortunately, both evaluations of $G_{1,2n+1}$ and $E^{(2)}_{1,2n+1} + 2^{2n+1}H_{1,2n+1}$ involve the common expression

$$\sum_{\ell=2}^{2n} 2^{\ell-1} S_{\ell,2n+2-\ell}.$$

Such a sum can be replaced by a linear combination of $S_{1,2n+1}$ and $\zeta(p)\zeta(q)$ with $p+q = 2n+2$ as we shall see in Section 4.

3. Explicit evaluation of $E^{(2)}_{1,2n+1} + 2^{2n+1}H_{1,2n+1}$

Our first attempt to evaluate $G_{1,2n+1}$ is unsuccessful, so we try another way to deal with $E_{1,2n+1}$ and $H_{1,2n+1}$ directly.

Proposition 3. For each positive integer n,

$$2^{-2n}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\} = \lambda(2n+1)\log 2 + \frac{1}{2}G_{1,2n+1} - \sum_{j=2}^{2n} 2^{-j} F_{2n+2-j,j}, \tag{3.1}$$

where for p, $q \geq 2$,

$$F_{p,q} = \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=0}^{k-1} \frac{1}{(2j+1)^p} = E^{(2)}_{p,q} - 2^{-p} S_{p,q}.$$

Proof. We begin with the difference

$$E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1} = \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{k-1} \frac{1}{2j+1}. \tag{3.2}$$

It follows that

$$2^{-(2n+1)}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\} = \sum_{k=1}^{\infty} \frac{1}{(2k)^{2n+1}} \sum_{j=0}^{k-1} \frac{1}{2j+1} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{(2k+2j+2)^{2n+1}(2j+1)} \tag{3.3}$$

with a simple change of variable $k = k' + j + 1$ in the summation. We need the partial fraction decomposition

$$\frac{1}{(x+2k+1)^{2n+1}x} = \frac{1}{(2k+1)^{2n+1}}\left\{\frac{1}{x} - \frac{1}{x+2k+1}\right\} - \frac{1}{(2k+1)(x+2k+1)^{2n+1}} - \sum_{\ell=2}^{2n}\frac{1}{(2k+1)^{\ell}(x+2k+1)^{2n+2-\ell}}. \tag{3.4}$$

Setting $x = 2j+1$ and letting j and k range over all non-negative integers, we get

$$2^{-(2n+1)}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+1}}\sum_{j=0}^{\infty}\left\{\frac{1}{2j+1} - \frac{1}{2j+2k+2}\right\} - \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(2k+1)(2j+2k+2)^{2n+1}} - \sum_{\ell=2}^{2n}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(2k+1)^{\ell}(2k+2n+2)^{2n+2-\ell}}.$$

The second term on the right-hand side is equal to

$$-2^{-(2n+1)}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\},$$

while the third term is equal to

$$-\sum_{\ell=2}^{2n}2^{-\ell}F_{2n+2-\ell,\ell}.$$

The inner summation of the first term is equal to

$$\frac{1}{2}\left\{-\frac{\Gamma'(1/2)}{\Gamma(1/2)} + \frac{\Gamma'(k+1)}{\Gamma(k+1)}\right\},$$

hence the whole term is equal to

$$\lambda(2n+1)\log 2 + \frac{1}{2}G_{1,2n+1}.$$

Thus our assertion follows.

In a similar way, we get

Proposition 4. *For each positive integer n,*

$$H_{1,2n+1} = \lambda(2n+2) + 2^{-(2n+1)}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\} - \frac{1}{2}G_{1,2n+1} - \sum_{\ell=2}^{2n}2^{-\ell}G_{l,2n+2-\ell}. \tag{3.5}$$

Proof. We begin with the definition of $H_{1,2n+1}$:

$$H_{1,2n+1} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \sum_{j=0}^{k} \frac{1}{2j+1},$$

which we rewrite as

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{(2k+2j+1)^{2n+1}(2j+1)} = \lambda(2n+2) + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{1}{(2k+2j+1)^{2n+1}(2j+1)}.$$

In order to evaluate the double series, we need the partial fraction decomposition:

$$\begin{aligned}\frac{1}{(2k+x)^{2n+1}x} = \frac{1}{(2k)^{2n+1}}\left\{\frac{1}{x} - \frac{1}{x+2k}\right\} - \frac{1}{2k(2k+x)^{2n}} \\ - \sum_{\ell=2}^{2n} \frac{1}{(2k)^{\ell}(2k+x)^{2n+2-\ell}}.\end{aligned}$$

Setting $x = 2j+1$ and letting j range over all non-negative integers and k range over all positive integers, we get

$$\begin{aligned}H_{1,2n+1} = \lambda(2n+2) + \sum_{k=1}^{\infty} \frac{1}{(2k)^{2n+1}} \sum_{j=0}^{\infty}\left\{\frac{1}{2j+1} - \frac{1}{2j+2k+1}\right\} \\ - \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{1}{2k(2k+2j+1)^{2n}} - \sum_{\ell=2}^{2n} \frac{1}{(2k)^{\ell}(2k+2j+1)^{2n+2-\ell}}.\end{aligned}$$

The second term in the right-hand side can be rewritten as

$$2^{-(2n+1)} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sum_{j=0}^{k-1} \frac{1}{2j+1},$$

which is equal to

$$2^{-(2n+1)}\left\{E^{(2)}_{1,2n+1} - \frac{1}{2}S_{1,2n+1}\right\}.$$

On the other hand, simply by the definition of $G_{p,q}$, the third term and final term are equal to $-\frac{1}{2}G_{1,2n+1}$ and $-\sum_{\ell=2}^{2n} 2^{-\ell}G_{\ell,2n+2-\ell}$, respectively. This proves the assertion.

Proof of Theorems 1, 2, and 3. Observe that for $p \geq 2$ and $q \geq 2$,

$$2^{-p}G_{p,q} + 2^{-p}F_{q,p} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{q}} \sum_{j=1}^{\infty} \frac{1}{(2j)^{p}} = 2^{-p}\zeta(p)\lambda(q). \tag{3.6}$$

So if we add formulas in Propositions 3 and 4 together, we get

$$2^{-(2n+1)}E^{(2)}_{1,2n+1} + H_{1,2n+1} = \lambda(2n+1)\log 2 + \lambda(2n+2) + 2^{-(2n+2)}S_{1,2n+1} - \sum_{\ell=2}^{2n} 2^{-\ell}\zeta(\ell)\lambda(2n+2-\ell). \tag{3.7}$$

Upon multiplying by 2^{2n+1} on both sides, we prove Theorem 2. By comparing two different expressions for $E^{(2)}_{1,2n+1} + 2^{2n+1}H_{1,2n+1}$ in Proposition 2 and Theorem 2, we get

$$\sum_{\ell=2}^{2n} 2^{\ell-1}S_{\ell,2n+2-\ell} = (2^{2n}-1)S_{1,2n} - 2^{2n}\zeta(2n+2) + \sum_{\ell=2}^{2n}\left(2^{2n+1-\ell} - \frac{1}{2}\right)\zeta(\ell)\zeta(2n+2-\ell), \tag{3.8}$$

which proves Theorem 3. Combining Theorem 3 and Proposition 1 then proves Theorem 1.

4. Relations among Euler sums of the same even weight

Some linear relations among Euler sums $S_{\ell,2n+2-\ell}$, $\ell = 2,3,\dots,2n$ can be obtained easily from the partial fraction decompositions of the rational functions

$$\frac{1}{x^{\ell}(x+1)^{2n+2-\ell}}, \quad \ell = 2,3,\dots,2n. \tag{4.1}$$

Here we illustrate the details. It is easy to see that

$$\frac{1}{x^2(x+1)^{2n}} = \frac{1}{x^2} - \frac{2n}{x} + \frac{2n}{x+1} + \frac{2n-1}{(x+1)^2} + \cdots + \frac{1}{(x+1)^{2n}}. \tag{4.2}$$

Replacing x by x/k, we get another identity ready to be used

$$\frac{1}{x^2(x+k)^{2n}} = \frac{1}{k^{2n}x^2} - \frac{2n}{k^{2n+1}}\left\{\frac{1}{x} - \frac{1}{x+k}\right\} + \sum_{\ell=2}^{2n}\frac{\ell-1}{k^{\ell}(x+k)^{2n+2-\ell}}. \tag{4.3}$$

As x and k both range over all positive integers and sum together, we get

$$S_{2,2n} - \zeta(2n+2) = \zeta(2)\zeta(2n) - 2nS_{1,2n+1} + \sum_{\ell=2}^{2n}(\ell-1)\left\{S_{\ell,2n+2-\ell} - \zeta(2n+2)\right\}, \tag{4.4}$$

which we rewrite as

$$S_{2,2n} + 2S_{3,2n-1} + 3S_{4,2n-2} + \cdots + (2n-1)S_{2n,2} = S_{2,2n} + \Delta(2), \tag{4.5}$$

where $\Delta(2)$ is a linear combination of $\zeta(2n+2)$, $\zeta(2)\zeta(2n)$ and $S_{1,2n+1}$.

On the other hand, from the partial fraction decomposition

$$\frac{1}{x^3(x+1)^{2n-1}} = \frac{1}{x^3} - \frac{2n-1}{x^2} + \frac{2n(2n-1)}{2}\left\{\frac{1}{x} - \frac{1}{x+1}\right\} - \sum_{\ell=2}^{2n-2} \frac{\ell(\ell-1)}{2}\frac{1}{(x+1)^\ell}, \tag{4.6}$$

we get the identity

$$\frac{1}{x^3(x+k)^{2n-1}} = \frac{1}{k^{2n-1}x^3} - \frac{2n-1}{k^{2n}x^2} + \frac{2n(2n-1)}{2}\frac{1}{k^{2n+1}}\left\{\frac{1}{x} - \frac{1}{x+k}\right\} - \sum_{\ell=2}^{2n-1} \frac{(2n-\ell)(2n-\ell-1)}{2}\frac{1}{k^{2n+2-\ell}(x+k)^\ell}. \tag{4.7}$$

Consequently, we have another relation

$$S_{3,2n-1} + 3S_{4,2n-2} + 6S_{5,2n-3} + \cdots + \frac{(2n-2)(2n-3)}{2}S_{2n,2} = -S_{3,2n-1} + \Delta(3),$$

where $\Delta(3)$ is a linear combination of $\zeta(2n+2)$, $\zeta(2)\zeta(2n)$, $\zeta(3)\zeta(2n-1)$ and $S_{1,2n+1}$.

Continuing this process, we get the following system of linear relations among $S_{\ell,2n+2-\ell}$, $\ell = 2, 3, \ldots, 2n$,

$$\begin{cases} S_{2,2n}+2S_{3,2n-1}+3S_{4,2n-2}+4S_{5,2n-3}+\cdots+(2n-1)S_{2n,2} = S_{2,2n}+\Delta(2) \\ S_{3,2n-1} + 3S_{4,2n-2} + 6S_{5,2n-3}+\cdots+\binom{2n-1}{2}S_{2n,2} = -S_{3,2n-1} + \Delta(3) \\ S_{4,2n-2} + 4S_{5,2n-3} + \cdots + \binom{2n-1}{3}S_{2n,2} = S_{4,2n-2} + \Delta(4) \\ \cdots\cdots\cdots\cdots\cdots \\ S_{2n-1,3} + (2n-1)S_{2n,2} = -S_{2n-1,3}+\Delta(2n-1) \\ S_{2n,2} = S_{2n,2} + \Delta(2n). \end{cases} \tag{4.8}$$

All adding together, we obtain

$$\sum_{\ell=2}^{2n}(2^{\ell-1}-1)S_{\ell,2n+2-\ell} = \sum_{\ell=2}^{2n}(-1)^\ell S_{\ell,2n+2-\ell} + \sum_{\ell=2}^{2n}\Delta(\ell).$$

Therefore,

$$\sum_{\ell=2}^{2n} 2^{\ell-1}S_{\ell,2n+2-\ell} = \sum_{\ell=2}^{2n} S_{\ell,2n+2-\ell} + \sum_{\ell=2}^{2n}(-1)^\ell S_{\ell,2n+2-\ell} + \sum_{\ell=2}^{2n}\Delta(\ell).$$

Note that

$$\sum_{\ell=2}^{2n} S_{\ell,2n+2-\ell} = \frac{1}{2}\sum_{\ell=2}^{2n} \zeta(\ell)\zeta(2n+2-\ell) - (2n-1)\zeta(2n)$$

and

$$\sum_{\ell=2}^{2n}(-1)^{\ell} S_{\ell,2n+2-\ell} = \frac{1}{2}\sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(\ell)\zeta(2n+2-\ell) + \begin{cases} \frac{1}{2}\zeta(2n+2), & \text{if } n \text{ is odd;} \\ -\frac{1}{2}\zeta(2n+2), & \text{if } n \text{ is even.} \end{cases}$$

So it is not a surprise that the sum

$$\sum_{\ell=2}^{2n} 2^{\ell-1} S_{\ell,2n+2-\ell}$$

can be expressed in terms of $\zeta(p)\zeta(q)$ with $p+q=2n+2$ and $S_{1,2n+1}$.

Remark. If we treat $S_{2,2n}, S_{3,2n-1}, \ldots, S_{2n,2}$ as $2n-1$ unknowns, the coefficient matrix of the system is given by

$$\begin{bmatrix} 0 & 2 & 3 & 4 & \ldots & \ldots & 2n-1 \\ 0 & 2 & 3 & 6 & \ldots & \ldots & \binom{2n-1}{2} \\ 0 & 0 & 0 & 4 & \ldots & \ldots & \binom{2n-1}{3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & 2 & 2n-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

The rank of this matrix is $n-2$. But still we have the relation

$$S_{p,q} + S_{q,p} = \zeta(p)\zeta(q) + \zeta(p+q), \tag{4.10}$$

and so

$$S_{n+1,n+1} = \frac{1}{2}\left\{\zeta^2(n+1) + \zeta(2n+2)\right\}. \tag{4.11}$$

So there are actually $n-1$ unknowns $S_{2,2n}, S_{3,2n-1}, \ldots, S_{n,n+2}$ to be determined from only $n-2$ independent linear equations. Of course, the result is undetermined. However, we are able to evaluate suitable linear combinations of a pair of sums. For example, when $n=5$, we have 4 unknowns $S_{2,10}, S_{3,9}, S_{4,8}, S_{5,7}$ subject to three conditions

$$\begin{cases} 9S_{2,10} + 6S_{3,9} + 4S_{4,8} + 2S_{5,7} = \Delta(2) \\ 36S_{2,10} + 26S_{3,9} + 18S_{4,8} + 9S_{5,7} = \Delta(3) \\ 84S_{2,10} + 56S_{3,9} + 35S_{4,8} + 16S_{5,7} = \Delta(4). \end{cases} \tag{4.12}$$

The above system is equivalent to

$$\begin{cases} 9S_{2,10} + 2S_{3,9} = \Delta'(2) \\ 16S_{3,9} + 9S_{4,8} = \Delta'(3) \\ 7S_{4,8} + 8S_{5,7} = \Delta'(4). \end{cases} \tag{4.13}$$

Consequently, we are able to express $9S_{2,10} + 2S_{3,9}$, $16S_{3,9} + 9S_{4,8}$ and $7S_{4,8} + 8S_{5,7}$ explicitly in terms of values of the Riemann zeta function at positive integers.

Exactly the same happens to $E^{(2)}_{1,3}$, $H_{1,3}$ and $G_{2,2}$. By Theorem 2, we have

$$E^{(2)}_{1,3} + 8H_{1,3} = \frac{1}{2}S_{1,3} + \frac{15}{2}\zeta(4) + 8\lambda(3)\log 2 - 2\zeta(2)\lambda(2)$$

or

$$E^{(2)}_{1,3} + 8H_{1,3} = \frac{35}{8}\zeta(4) + 7\zeta(3)\log 2. \tag{4.14}$$

Also from Proposition 4, we get

$$E^{(2)}_{1,3} - 8H_{1,3} - 2G_{2,2} = -\frac{5}{4}\zeta(4) - 7\zeta(3)\log 2. \tag{4.15}$$

With only (4.14) and (4.15) at our disposal, we are unable to determine $E^{(2)}_{1,3}$, $H_{1,3}$ and $G_{2,2}$ individually.

The above discussion indicates the indeterminacy nature of Ramanujan's constant $G(1)$, which is due to the lack of linearly independent relations among Euler sums of the same even weight.

5. A further generalization

For a pair of positive integers p and q with $q \geq 2$ and another pair of positive integers a and b, we define new sums

$$G_{p,q}(a,b) = \sum_{k=0}^{\infty} \frac{1}{(ak+b)^q} \sum_{j=1}^{k} \frac{1}{j^p} \tag{5.1}$$

and

$$H_{p,q}(a,b) = \sum_{k=0}^{\infty} \frac{1}{(ak+b)^q} \sum_{j=0}^{k} \frac{1}{(aj+b)^p}. \tag{5.2}$$

When $a = 2$, $b = 1$ and $p + q$ is odd, the evaluations of $G_{p,q}(2,1)$ and $H_{p,q}(2,1)$ are already known [6]. Here we shall give the evaluations of $G_{1,n}(a,b)$ and

$$E^{(a)}_{1,n} + a^n \sum_{b=1}^{a-1} H_{1,n}(a,b)$$

through relations among $G_{p,q}(a,b)$, $H_{p,q}(a,b)$ and $E^{(a)}_{p,q}$.

Theorem 4. *For positive integers $n \geq 2$ and a pair of positive integers a and b, we have*

$$\begin{aligned} G_{1,n}(a,b) = {} & \frac{n}{2a^n}\zeta(n+1,\frac{b}{a}) + \frac{1}{a^n}\zeta(n,\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)} + \gamma\right\} \\ & -\frac{1}{2a^n}\sum_{\ell=2}^{n-1}\zeta(\ell,\frac{b}{a})\zeta(n+1-\ell,\frac{b}{a}). \end{aligned} \tag{5.3}$$

Proof. Right from the definition (5.1), we have

$$a^{-1}G_{1,n}(a,b)=\sum_{k=0}^{\infty}\frac{1}{(ak+b)^n}\sum_{j=1}^{k}\frac{1}{aj}=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(ak+aj+b)^n aj}.$$

We need the following partial fraction decomposition

$$\frac{1}{(x+\alpha)^n x}=\frac{1}{\alpha^n}\left\{\frac{1}{x}-\frac{1}{x+\alpha}\right\}-\frac{1}{\alpha(x+\alpha)^n}-\sum_{\ell=2}^{n-1}\frac{1}{\alpha^\ell(x+\alpha)^{n+1-\ell}}.$$

Set $x=aj$ and $\alpha=ak+b$. Then letting j range over all positive integers, k range over all non-negative integers and summing together, we get

$$\begin{aligned}a^{-1}G_{1,n}(a,b)=&\sum_{k=0}^{\infty}\frac{1}{(ak+b)^n}\sum_{j=1}^{\infty}\left\{\frac{1}{aj}-\frac{1}{ak+aj+b}\right\}\\&-\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(ak+b)(ak+aj+b)^n}\\&-\sum_{\ell=2}^{n-1}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(ak+b)^\ell(ak+aj+b)^{n+1-\ell}}.\end{aligned}$$

The first term is equal to

$$\frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{(ak+b)^n}\left\{\frac{\Gamma'(k+1+b/a)}{\Gamma(k+1+b/a)}-\frac{\Gamma'(1)}{\Gamma(1)}\right\},$$

which we rewrite as

$$\sum_{k=0}^{\infty}\frac{1}{(ak+b)^n}\sum_{j=0}^{k}\frac{1}{aj+b}+\frac{1}{a}\sum_{k=0}^{\infty}\frac{1}{(ak+b)^n}\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)}+\gamma\right\}$$

or simply as

$$H_{1,n}(a,b)+\frac{1}{a^{n+1}}\zeta(n,\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)}+\gamma\right\}.$$

The second term is equal to the negative of

$$H_{1,n}(a,b)-\frac{1}{a^{n+1}}\zeta(n+1,\frac{b}{a})$$

and the third term is equal to the negative of

$$\sum_{\ell=2}^{n-1}\left\{H_{\ell,n+1-\ell}(a,b)-\frac{1}{a^{n+1}}\zeta(n+1,\frac{b}{a})\right\}.$$

Consequently, we have

$$G_{1,n}(a,b) = \frac{n-1}{a^n}\zeta(n+1,\frac{b}{a}) + \frac{1}{a^n}\zeta(n,\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)}+\gamma\right\} - a\sum_{\ell=2}^{n-1} H_{\ell,n+1-\ell}(a,b).$$

Our assertion then follows from

$$\sum_{\ell=2}^{n-1} H_{\ell,n+1-\ell}(a,b) = \frac{1}{2}\sum_{\ell=2}^{n-1}\{H_{\ell,n+1-\ell}(a,b)+H_{n+1-\ell,\ell}(a,b)\} = \frac{1}{2a^{n+1}}\sum_{\ell=2}^{n-1}\left\{\zeta(\ell,\frac{b}{a})\zeta(n+1-\ell,\frac{b}{a})+\zeta(n+1,\frac{b}{a})\right\}.$$

Theorem 5. For positive integers n and a with $n \geq 2$, $a \geq 2$, we have

$$E_{1,n}^{(a)} + a^n\sum_{b=1}^{a-1} H_{1,n}(a,b) = a^{-1}S_{1,n} + \left(a^n - \frac{1}{a}\right)\zeta(n+1) - \sum_{b=1}^{a-1}\zeta(n,1-\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)}+\gamma\right\} - \sum_{b=1}^{a-1}\sum_{\ell=2}^{n-1} a^{-1}\zeta(\ell)\zeta(n+1-\ell,\frac{b}{a}). \tag{5.4}$$

Proof. At first, we have

$$H_{1,n}(a,b) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(ak+aj+b)^n(aj+b)} = a^{-(n+1)}\zeta(n+1,\frac{b}{a}) + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(ak+aj+b)^n(aj+b)}.$$

With the help of partial fraction decomposition, the second term is equal to

$$\sum_{k=1}^{\infty}\frac{1}{(ak)^n}\sum_{j=0}^{\infty}\left\{\frac{1}{aj+b}-\frac{1}{ak+aj+b}\right\} - \sum_{\ell=1}^{n-1}\frac{1}{(ak)^{\ell}(ak+aj+b)^{n+1-\ell}},$$

or

$$a^{-n}\sum_{k=1}^{\infty}\frac{1}{k^n}\sum_{j=0}^{k-1}\frac{1}{aj+b} - \sum_{\ell=1}^{n-1} a^{-\ell}G_{\ell,n+1-\ell}(a,b).$$

Let b range over $1, 2, \ldots, a-1$ and sum together, we get

$$\sum_{b=1}^{a-1} H_{1,n}(a,b) = \sum_{b=1}^{a-1} a^{-(n+1)}\zeta(n+1, \frac{b}{a}) + a^{-n}\left\{E_{1,n}^{(a)} - \frac{1}{a}S_{1,n}\right\} - \sum_{b=1}^{a-1}\sum_{\ell=1}^{n-1} a^{-\ell} G_{\ell,n+1-\ell}(a,b).$$

On the other hand, we have

$$\begin{aligned} a^{-n}\left\{E_{1,n}^{(a)} - a^{-1}S_{1,n}\right\} &= \sum_{k=1}^{\infty} \frac{1}{(ak)^n}\left\{\sum_{j=1}^{ak}\frac{1}{j} - \sum_{j=1}^{k}\frac{1}{aj}\right\} \\ &= \sum_{b=1}^{a-1}\sum_{k=1}^{\infty}\frac{1}{(ak)^n}\sum_{j=0}^{k-1}\frac{1}{aj+b} \\ &= \sum_{b=1}^{a-1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(ak+aj+a)^n(aj+b)}. \end{aligned}$$

As before, we employ partial fraction decomposition to get

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(ak+aj+a)^n(aj+b)} = a^{-1}G_{1,2n}(a, a-b) - a^{-n}\zeta(n, 1-\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)} + \gamma\right\} - \sum_{\ell=1}^{n-1}\sum_{k=1}^{\infty}\frac{1}{(ak)^{n+1-\ell}}\sum_{j=0}^{k-1}\frac{1}{(aj+a-b)^{\ell}}.$$

So it follows that

$$2a^{-n}\left\{E_{1,n}^{(a)} - a^{-1}S_{1,n}\right\} = a^{-1}\sum_{b=1}^{a-1} G_{1,2n}(a, a-b) - a^{-n}\sum_{b=1}^{a-1}\zeta(n, 1-\frac{b}{a})\left\{\frac{\Gamma'(b/a)}{\Gamma(b/a)} + \gamma\right\} - \sum_{\ell=2}^{n-1} a^{-\ell}\left\{E_{n+1-\ell,\ell}^{(a)} - a^{-(n+1)+\ell}S_{n+1-\ell,\ell}\right\}.$$

Our assertion then follows from

$$\left\{E_{p,q}^{(a)} - a^{-p}S_{p,q}\right\} + \sum_{b=1}^{a-1} a^{-q}G_{q,p}(a,b) = \sum_{b=1}^{a-1} a^{-(p+q)}\zeta(p)\zeta(q, \frac{b}{a}).$$

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