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A REFINEMENT OF THE GRÜSS INEQUALITY AND APPLICATIONS

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Abstract. A sharp refinement of the Grüss inequality in the general setting of measurable spaces and abstract Lebesgue integrals is proven. Some consequential particular inequalities are mentioned.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu-$ a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R}, f$ is μ-measurable and $\int_\Omega w(x)|f(x)|d\mu(x) < \infty\}$. Assume $\int_\Omega w(x)d\mu(x) > 0$.

If $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} T_w(f,g) := & \frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x)f(x)g(x)d\mu(x) \\ & - \frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x)f(x)d\mu(x) \\ & \times \frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x)g(x)d\mu(x). \end{aligned} \tag{1.1}$$

The following result is known in the literature as the Grüss inequality

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for $\mu-$ a.e. $x \in \Omega$.

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The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega = \{1, \ldots, n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i y_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \cdot \frac{1}{W_n} \sum_{i=1}^{n} w_i y_i \right| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

provided $\gamma \leq x_i \leq \Gamma$, $\delta \leq y_i \leq \Delta$ for each $i \in \{1, \ldots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^{n} w_i > 0$.

The following result was proved in Cheng and Sun [4].

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\delta \leq g(x) \leq \Delta$ for some constants δ, Δ for all $x \in [a, b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right|$$
$$\leq \frac{\Delta - \delta}{2} \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y)dy \right| dx. \tag{1.5}$$

They used the result (1.5) to obtain perturbed trapezoidal rules.

In the current paper we obtain bounds for $|T_w(f, g)|$ under the general setting expressed in (1.1). A bound which is shown to be *sharp* is obtained in Section 2. The sharpness of (1.5) was not demonstrated in [4]. Sharp results were obtained for a perturbed interior point rule (Ostrowski-Grüss) inequalities in Cheng [3]. Some particular instances of the results in Section 2 are investigated in Sections 4 and 5, recapturing earlier work. Results are presented in Section 3, for Lebesgue measurable functions and for a discrete weighted Čebyšev functional involving $n-$tuples.

2. An integral inequality

With the assumptions as presented in the Introduction and if $f \in L_w(\Omega, \mu)$ then we may define

$$\begin{aligned} D_w(f) &:= D_{w,1}(f) \\ &:= \frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x) \\ &\quad \times \left| f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y)f(y)d\mu(y) \right| d\mu(x). \end{aligned} \tag{2.1}$$

The following fundamental result holds.

Theorem 2. *Let $w, f, g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ- a.e. on Ω and $\int_\Omega w(y)d\mu(y) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \text{ for } \mu- \text{ a.e. } x \in \Omega, \tag{2.2}$$

then we have the inequality

$$|T_w(f,g)| \le \frac{1}{2}(\Delta - \delta) D_w(f). \tag{2.3}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Proof. Obviously, we have

$$\begin{aligned} T_w(f,g) = &\frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x) \\ &\times \Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega f(y)w(y)d\mu(y)\Big) g(x)d\mu(x). \end{aligned} \tag{2.4}$$

Consider the measurable subsets Ω_+ and Ω_-, of Ω, defined by

$$\Omega_+ := \left\{ x \in \Omega \,\middle|\, f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y) f(y) d\mu(y) \ge 0 \right\}$$

and

$$\Omega_- := \left\{ x \in \Omega \,\middle|\, f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y) f(y) d\mu(y) < 0 \right\}.$$

Obviously, $\Omega = \Omega_+ \cup \Omega_-$, $\Omega_+ \cap \Omega_- = \emptyset$ and if we define

$$\begin{aligned} I_+(f,g,w) := &\frac{1}{\int_\Omega w(y)d\mu(y)} \int_{\Omega_+} w(x) \\ &\times \Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y)f(y)d\mu(y)\Big) g(x)d\mu(x) \end{aligned}$$

and

$$\begin{aligned} I_-(f,g,w) := &\frac{1}{\int_\Omega w(y)d\mu(y)} \int_{\Omega_-} w(x) \\ &\times \Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y)f(y)d\mu(y)\Big) g(x)d\mu(x) \end{aligned}$$

then we have

$$T_w(f,g) = I_+(f,g,w) + I_-(f,g,w). \tag{2.5}$$

Since $-\infty < \delta \le g(x) \le \Delta < \infty$ for μ – a.e. $x \in \Omega$ and $w(x) \ge 0$ for μ – a.e. $x \in \Omega$, we may write:

$$\begin{aligned} I_+(f,g,w) \le &\frac{\Delta}{\int_\Omega w(y)d\mu(y)} \int_{\Omega_+} w(x) \\ &\times \Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega f(y)w(y)d\mu(y)\Big) d\mu(x) \end{aligned} \tag{2.6}$$

and

$$I_{-}(f,g,w) \leq \frac{\delta}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_{-}} w(x) \times \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x). \tag{2.7}$$

Since

$$\begin{aligned} 0 &= \int_{\Omega} w(x) \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x) \\ &= \int_{\Omega_{+}} w(x) \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x) \\ &\quad + \int_{\Omega_{-}} w(x) \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x) \end{aligned}$$

we get

$$\begin{aligned} &\int_{\Omega_{-}} w(x) \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x) \\ &\quad = - \int_{\Omega_{+}} w(x) \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \Big) d\mu(x) \end{aligned}$$

and thus, from (2.7), we deduce

$$I_{-}(f,g,w) \leq \frac{-\delta}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_{+}} w(x) \times \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y) \Big) d\mu(x). \tag{2.8}$$

Consequently, by adding (2.6) with (2.8), we deduce

$$T_w(f,g) \leq \frac{\Delta - \delta}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_{+}} w(x) \times \Big(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y) \Big) d\mu(x). \tag{2.9}$$

On the other hand,

$$\begin{aligned} &\int_{\Omega} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \right| d\mu(x) \\ &\quad = \int_{\Omega_{+}} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \right| d\mu(x) \\ &\qquad + \int_{\Omega_{-}} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y) \right| d\mu(x) \end{aligned}$$

$$= \int_{\Omega_+} w(x)\Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega f(y)w(y)d\mu(y)\Big)d\mu(x)$$
$$- \int_{\Omega_-} w(x)\Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega f(y)w(y)d\mu(y)\Big)d\mu(x)$$
$$= 2\int_{\Omega_+} w(x)\Big(f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega f(y)w(y)d\mu(y)\Big)d\mu(x),$$

and thus, by (2.9) we deduce

$$T_w(f,g) \le \frac{1}{2}(\Delta - \delta)\frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega w\,(x) \times \left| f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega f(y)w(y)d\mu\,(y)\right| d\mu(x). \tag{2.10}$$

Now, if we write the inequality (2.10) for $-f$ instead of f and taking into account that $T_w(-f,g) = -T_w(f,g)$, we deduce

$$-T(f,g) \le \frac{1}{2}(\Delta - \delta)\frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega w\,(x) \times \left| f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)}\int_\Omega f(y)w(y)d\mu(y)\right| d\mu(x), \tag{2.11}$$

giving the desired inequality (2.3).

To prove the sharpness of the constant $\frac{1}{2}$, assume that (2.3) holds for $\Omega = [a,b]$ and $w \equiv 1$, with a constant $C > 0$. That is,

$$|T(f,g)| \le C(\Delta - \delta)\frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f\,(y)\,dy\right| dx, \tag{2.12}$$

where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g\,(x)\,dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx$$

and the integral $\int_a^b$ is the usual Lebesgue integral on $[a,b]$.

Choose in (2.12) $g = f$ and $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 \text{ if } & x \in \left[a, \frac{a+b}{2}\right], \\ 1 \quad \text{if } & x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then, obviously,

$$T(f,f) = \frac{1}{b-a}\int_a^b f^2(x)dx - \Big(\frac{1}{b-a}\int_a^b f(x)dx\Big)^2 = 1,$$

$$D(f) = \frac{1}{b-a}\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(y)dy \right| dx = 1,$$
$$\delta = -1, \quad \Delta = 1,$$

and by (2.12) we get $2C \geq 1$ giving $C \geq \frac{1}{2}$.

For $f \in L_{p,w}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w(x) \left| f(x) \right|^p d\mu(x) < \infty \right\}$, $p \geq 1$ we may also define

$$\begin{aligned} D_{w,p}(f) &:= \left[\frac{1}{\int_\Omega w(x)d\mu(x)} \int_\Omega w(x) \right. \\ &\quad \left. \times \left| f(x) - \frac{1}{\int_\Omega w(y)d\mu(y)} \int_\Omega w(y)f(y)d\mu(y) \right|^p d\mu(x) \right]^{\frac{1}{p}} \\ &= \frac{\left\| f - \frac{1}{\int_\Omega wd\mu} \int_\Omega wfd\mu \right\|_{\Omega,p}}{\left[\int_\Omega w(x)d\mu\,(x) \right]^{\frac{1}{p}}} \end{aligned} \tag{2.13}$$

where $\|\cdot\|_{\Omega,p}$ is the usual $p-$norm on $L_{p,w}(\Omega, \mathcal{A}, \mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_\Omega w \left| h \right|^p d\mu \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Using Hölder's inequality we get

$$D_{w,1}(f) \leq D_{w,p}(f) \text{ for } p \geq 1, \; f \in L_{p,w}(\Omega, \mathcal{A}, \mu); \tag{2.14}$$

and, in particular for $p = 2$

$$D_{w,1}(f) \leq D_{w,2}(f) = \left[\frac{\int_\Omega wf^2 d\mu}{\int_\Omega wd\mu} - \left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)^2 \right]^{\frac{1}{2}}, \tag{2.15}$$

if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$.

For $f \in L_\infty(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \; \|f\|_{\Omega,\infty} := ess \sup_{x\in\Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \frac{1}{\int_\Omega wd\mu} \int_\Omega wfd\mu \right\|_{\Omega,\infty}. \tag{2.16}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem* 2, *we have*

$$\begin{aligned} |T_w(f,g)| &\leq \frac{1}{2}(\Delta - \delta)D_w(f) \\ &\leq \frac{1}{2}(\Delta - \delta)D_{w,p}(f) \quad \text{if } f \in L_p(\Omega, \mathcal{A}, \mu), \; 1 < p < \infty; \end{aligned}$$

$$\le \frac{1}{2}(\Delta-\delta)\left\|f-\frac{1}{\int_\Omega w d\mu}\int_\Omega wf d\mu\right\|_{\Omega,\infty} \quad \text{if } f\in L_\infty(\Omega,\mathcal{A},\mu). \tag{2.17}$$

Remark 1. The inequalities in (2.17) are in order of increasing coarseness. If we assume that $-\infty<\gamma\le f(x)\le\Gamma<\infty$ for μ − a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ we have for $p=2$

$$\left[\frac{\int_\Omega wf^2 d\mu}{\int_\Omega w d\mu}-\Big(\frac{\int_\Omega wf d\mu}{\int_\Omega w d\mu}\Big)^2\right]^{\frac{1}{2}}\le\frac{1}{2}(\Gamma-\gamma). \tag{2.18}$$

By (2.17), we deduce the following sequence of inequalities

$$\begin{aligned}|T_w(f,g)| &\le \frac{1}{2}(\Delta-\delta)\frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f-\frac{1}{\int_\Omega w d\mu}\int_\Omega wf d\mu\right| d\mu\\ &\le \frac{1}{2}(\Delta-\delta)\left[\frac{\int_\Omega wf^2 d\mu}{\int_\Omega w d\mu}-\Big(\frac{\int_\Omega wf d\mu}{\int_\Omega w d\mu}\Big)^2\right]^{\frac{1}{2}}\\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma) \end{aligned}\tag{2.19}$$

for $f,g:\Omega\to\mathbb{R}$, μ − measurable functions and so that $-\infty<\gamma\le f(x)<\Gamma<\infty$, $-\infty<\delta\le g(x)\le\Delta<\infty$ for μ − a.e. $x\in\Omega$. Thus, the inequality (2.19) is a refinement of Grüss' inequality (1.2).

It is well known that if $f\in L_{2,w}(\Omega,\mathcal{A},\mu)$, then the following Schwarz's type inequality holds:

$$\frac{1}{\int_\Omega w d\mu}\int_\Omega wf^2 d\mu\ge\Big(\frac{1}{\int_\Omega w d\mu}\int_\Omega wf d\mu\Big)^2. \tag{2.20}$$

Using the above results, we may point out the following counterpart result.

Proposition 1. *Assume that the μ-measurable function $f:\Omega\to\mathbb{R}$ satisfies the assumption:*

$$-\infty<\gamma\le f(x)\le\Gamma<\infty \quad \textit{for a.e. } x\in\Omega. \tag{2.21}$$

Then one has the inequality

$$\begin{aligned}0&\le\frac{1}{\int_\Omega w d\mu}\int_\Omega wf^2 d\mu-\Big(\frac{1}{\int_\Omega w d\mu}\int_\Omega wf d\mu\Big)^2\\ &\le\frac{1}{2}(\Gamma-\gamma)\frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f-\frac{1}{\int_\Omega w d\mu}\int_\Omega wf d\mu\right| d\mu\\ &\Big(\le\frac{1}{4}(\Gamma-\gamma)^2\Big).\end{aligned}\tag{2.22}$$

The constant $\frac{1}{2}$ is sharp.

The proof follows by the inequality (2.3) for $g = f$.

The following proposition also holds.

Proposition 2. *Assume that the measurable functions* $f, g : \Omega \to \mathbb{R}$ *satisfy* (1.3) (*the condition in Grüss' inequality*). *Then*

$$\begin{aligned} |T_w(f,g)| &\le \frac{1}{2}\Big[(\Gamma-\gamma)(\Delta-\delta)\Big]^{\frac{1}{2}}\Big[D_w(f)D_w(g)\Big]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma). \end{aligned} \tag{2.23}$$

The constant $\frac{1}{2}$ *in the first inequality and* $\frac{1}{4}$ *in the second inequality are sharp.*

Proof. By (2.19) we have

$$|T_w(f,g)| \le \frac{1}{2}(\Delta-\delta)D_w(f)$$

and

$$|T_w(f,g)| \le \frac{1}{2}(\Gamma-\gamma)D_w(g)$$

from which, by multiplication, gives the first part of (2.23).

The second part and the sharpness of the constants are obvious.

3. Some particular inequalities

The following particular inequalities are of interest.

1. Let $w, f, g : [a,b] \to \mathbb{R}$ be Lebesgue measurable functions with $w \ge 0$ a.e. on $[a,b]$ and $\int_a^b w(y)dy > 0$. If $f, g, fg \in L_w[a,b]$, where

$$L_w[a,b] := \left\{ f : [a,b] \to \mathbb{R} \,\middle|\, \int_a^b w(x)\,|f(x)|\,dx < \infty \right\}$$

and

$$-\infty < \delta \le g(x) \le \Delta < \infty \quad \text{for a.e. } x \in [a,b], \tag{3.1}$$

then we have the inequalities

$$\begin{aligned} &\left| \frac{1}{\int_a^b w(x)dx}\int_a^b w(x)f(x)g(x)dx \right. \\ &\qquad \left. - \frac{1}{\int_a^b w(x)dx}\int_a^b w(x)f(x)dx \cdot \frac{1}{\int_a^b w(x)dx}\int_a^b w(x)g(x)dx \right| \\ &\quad \le \frac{1}{2}(\Delta-\delta)\frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)\left| f(x) - \frac{1}{\int_a^b w(y)dy}\int_a^b w(y)f(y)\,dy \right| dx \end{aligned}$$

$$\le \frac{1}{2}(\Delta-\delta)\left[\frac{\int_a^b w(x)\left|f(x)-\frac{1}{\int_a^b w(y)dy}\int_a^b w(y)f\left(y\right)dy\right|^p dx}{\int_a^b w(x)dx}\right]^{\frac{1}{p}}$$

$$\text{if} \quad f\in L_{p,w}\left[a,b\right], \quad 1<p<\infty,$$

$$\le \frac{1}{2}(\Delta-\delta)ess\sup_{x\in[a,b]}\left|f(x)-\frac{1}{\int_a^b w(y)dy}\int_a^b w(y)f(y)dy\right| \text{ if } f\in L_\infty\left[a,b\right]. \quad (3.2)$$

The constant $\frac{1}{2}$ is sharp in the first inequality in (3.2).
The following counterpart of Schwarz's inequality holds

$$0\le \frac{1}{\int_a^b w(y)dy}\int_a^b w(x)f^2(x)dx-\Big(\frac{1}{\int_a^b w(y)dy}\int_a^b w(x)f(x)dx\Big)^2$$

$$\le \frac{1}{2}(\Delta-\gamma)\frac{1}{\int_a^b w(y)dy}\int_a^b w(x)\left|f(x)-\frac{1}{\int_a^b w(y)dy}\int_a^b w(y)f(y)dy\right|dx$$

$$\Big(\le \frac{1}{4}(\Gamma-\gamma)^2\Big), \quad (3.3)$$

provided $-\infty<\gamma\le f(x)\le\Gamma<\infty$ for a.e. $x\in[a,b]$. The constant $\frac{1}{2}$ is sharp.
If $w(x)=1$, $x\in[a,b]$, then we recapture the result in [4] as depicted here by (1.5).

2. Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ be $n-$tuples of real numbers with $p_i\ge 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$. If

$$b\le b_i\le B, \quad i\in\{1,\dots,n\}, \quad (3.4)$$

then one has the inequality

$$\left|\sum_{i=1}^n p_ia_ib_i-\sum_{i=1}^n p_ia_i\cdot\sum_{i=1}^n p_ib_i\right|\le\frac{1}{2}(B-b)\sum_{i=1}^n p_i\left|a_i-\sum_{j=1}^n p_ja_j\right|$$

$$\le\frac{1}{2}(B-b)\left[\sum_{i=1}^n p_i\left|a_i-\sum_{j=1}^n p_ja_j\right|^p\right]^{\frac{1}{p}} \quad \text{if} \quad 1<p<\infty$$

$$\le\frac{1}{2}(B-b)\max_{i=\overline{1,n}}\left|a_i-\sum_{j=1}^n p_ja_j\right|. \quad (3.5)$$

The constant $\frac{1}{2}$ is sharp in the first inequality.
If $p_i=1$, $i\in\{1,\dots n\}$, the following unweighted inequality may be stated

$$0\le\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i$$

$$\le \frac{1}{2}(B-b)\frac{1}{n}\sum_{i=1}^{n}\left|a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right|$$
$$\le \frac{1}{2}(B-b)\left(\frac{1}{n}\sum_{i=1}^{n}\left|a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^p\right)^{\frac{1}{p}}$$
$$\le \frac{1}{2}(B-b)\max_{i=\overline{1,n}}\left|a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right|. \tag{3.6}$$

The following counterpart of Schwarz's inequality also holds

$$0 \le \sum_{i=1}^{n}p_ia_i^2-(\sum_{i=1}^{n}p_ia_i)^2 \le \frac{1}{2}(A-a)\sum_{i=1}^{n}p_i\left|a_i-\sum_{j=1}^{n}p_ja_j\right| \Big(\le \frac{1}{4}(A-a)^2\Big), \tag{3.7}$$

provided $a \le a_i \le A$ for each $i \in \{1,\dots,n\}$ and $\sum_{i=1}^{n}p_i = 1$. The constant $\frac{1}{2}$ is sharp.

4. Applications for Ostrowski's inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ such that $\varphi' \in L_\infty[a,b]$, then the following inequality is known in the literature as Ostrowski's inequality

$$\left|\varphi(x)-\frac{1}{b-a}\int_a^b\varphi(t)\,dt\right| \le \left[\frac{1}{4}+\Big(\frac{x-\frac{a+b}{2}}{b-a}\Big)^2\right]\|\varphi'\|_\infty(b-a), \quad x\in[a,b], \tag{4.1}$$

where $\|\varphi'\|_\infty := ess\sup_{\alpha\in[a,b]}|\varphi'(x)|$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact, as mentioned in [1], may be accomplished by the use of the Montgomery identity

$$\varphi(x)=\frac{1}{b-a}\int_a^b\varphi(t)dt+\frac{1}{b-a}\int_a^bK(x,t)\varphi'(t)\,dt, \tag{4.2}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ is defined by

$$K(x,t) := \begin{cases} t-a \text{ if } & a\le t\le x \\ t-b \text{ if } & a\le x<t\le b. \end{cases} \tag{4.3}$$

We will now use the unweighted version of the inequality (3.2), namely, (1.5) (obtained by Cheng and Sun [4]) to procure the next result concerning a perturbed version of Ostrowski's inequality (4.1).

The following result also obtained by Cheng [3] is recaptured in a simpler manner. A weighted version of this result was obtained by Roumeliotis [5].

Theorem 3. *Assume that* $\varphi : [a,b] \to \mathbb{R}$ *is an absolutely continuous function on* $[a,b]$ *such that* $\varphi' : [a,b] \to \mathbb{R}$ *satisfies the condition*

$$-\infty < \gamma \leq \varphi'(x) \leq \Gamma < \infty \quad \text{for} \quad \text{a.e. } x \in [a,b]. \tag{4.4}$$

Then we have the inequality

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)\,dt - \left(x - \frac{a+b}{2}\right)[\varphi; a, b]\right| \leq \frac{1}{8}(b-a)(\Gamma-\gamma) \tag{4.5}$$

for any $x \in [a,b]$, *where* $[\varphi; a, b] = \frac{\varphi(b)-\varphi(a)}{b-a}$ *is the divided difference. The constant* $\frac{1}{8}$ *is best possible.*

Proof. We apply inequality (3.1) for the choices $w(t) = 1$, $f(t) = K(x,t)$ defined by (4.3), $g(t) = \varphi'(t)$, $t \in [a,b]$ to get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b K(x,t)\varphi'(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)dt \cdot \frac{1}{b-a}\int_a^b \varphi'(t)dt\right| \\ &\quad \leq \frac{1}{2}(\Gamma-\gamma)\cdot\frac{1}{b-a}\int_a^b\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right|dt. \end{aligned} \tag{4.6}$$

We obviously have,

$$\frac{1}{b-a}\int_a^b K(x,t)dt = x - \frac{a+b}{2}$$

and

$$\frac{1}{b-a}\int_a^b \varphi'(t)dt = \frac{\varphi(b)-\varphi(a)}{b-a}.$$

Also

$$\begin{aligned} I(x) &:= \frac{1}{b-a}\int_a^b \left|K(x,t) - \left(x - \frac{a+b}{2}\right)\right| dt \\ &= \frac{1}{b-a}\left[\int_a^x \left|t - a - x + \frac{a+b}{2}\right| dt + \int_x^b \left|t - b - x + \frac{a+b}{2}\right| dt\right] \\ &= \frac{1}{b-a}\left[\int_a^x \left|t - x + \frac{b-a}{2}\right| dt + \int_x^b \left|t - x - \frac{b-a}{2}\right| dt\right]. \end{aligned}$$

Straight forward substitution of $u = t - x + \frac{b-a}{2}$ and $v = t - x - \frac{b-a}{2}$ gives

$$I(x) = \frac{1}{b-a}\left[\int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}} |u|\,du + \int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x} |v|\,dv\right]$$
$$= \frac{1}{b-a}\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |u|\,du = \frac{2}{b-a}\int_{0}^{\frac{b-a}{2}} udu = \frac{b-a}{4}.$$

Substitution of the above into (4.6) produces (4.5). The sharpness of the constant was proved in [3].

5. Application for the Generalised trapezoid inequality

If $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' \in L_\infty [a,b]$, then the following inequality is known as the generalised trapezoid inequality

$$\left|(x-a)\varphi(a) + (b-x)\varphi(b) - \int_a^b \varphi(t)dt\right| \le \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|\varphi'\|_\infty \tag{5.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

A simple proof of this fact is accomplished by using the identity [2]

$$\int_a^b \varphi(t)dt = (x-a)\varphi(a) + (b-x)\varphi(b) + \int_a^b (x-t)\,\varphi'(t)dt. \tag{5.2}$$

Utilising the inequality (3.1) we may point out the following perturbed version of (5.1).

Theorem 4. *Assume that $\varphi : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ so that $\varphi' : [a,b] \to \mathbb{R}$ satisfies the condition (4.4). Then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b \varphi(t)dt - \left[\left(\frac{x-a}{b-a}\right)\varphi(a) + \left(\frac{b-x}{b-a}\right)\varphi(b)\right] - \left(x - \frac{a+b}{2}\right)[\varphi; a, b]\right| \le \frac{1}{8}(b-a)(\Gamma - \gamma) \tag{5.3}$$

for any $x \in [a,b]$, where $[\varphi; a, b]$ is the divided difference. The constant $\frac{1}{8}$ is sharp.

Proof. We apply inequality (3.2) for the choices $f(t) = (x-t)$, $g(t) = \varphi'(t)$, $w(t) = 1$, $t \in [a,b]$, to get

$$\left|\frac{1}{b-a}\int_a^b (x-t)\varphi'(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)dt \cdot \frac{1}{b-a}\int_a^b \varphi'(t)dt\right|$$

$$\leq \frac{1}{2}(\Gamma - \gamma)\frac{1}{b-a}\int_a^b \left|(x-t) - \frac{1}{b-a}\int_a^b (x-s)\,ds\right| dt. \tag{5.4}$$

Since

$$\frac{1}{b-a}\int_a^b (x-t)dt = \Big(x - \frac{a+b}{2}\Big),$$
$$\frac{1}{b-a}\int_a^b \varphi'(t)dt = \frac{\varphi(b)-\varphi(a)}{b-a} = [\varphi; a, b]$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b \left|(x-t) - \frac{1}{b-a}\int_a^b (x-s)ds\right| dt &= \frac{1}{b-a}\int_a^b \left|x - t - x + \frac{a+b}{2}\right| dt \\ &= \frac{1}{b-a}\int_a^b \left|t - \frac{a+b}{2}\right| dt \\ &= \frac{b-a}{4},\end{aligned}$$

from (5.4) we deduce the desired inequality (5.3).

The sharpness of the constant may be shown on choosing $x = \frac{a+b}{2}$ and $\varphi(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a, b]$. We omit the details.

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