NEW PROPERTIES OF THE JUNG-KIM-SRIVASTAVA INTEGRAL OPERATORS

B. A. FRASIN

Abstract. The object of the present paper is to prove new subordinations results of analytic functions defined by two integral operators $P_\alpha$ and $Q_\alpha$. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$  \hspace{1cm} (1.1)

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. For the functions $f$ and $g$ in $\mathcal{A}$, we say that $f$ is subordinate to $g$ in $\mathcal{U}$, and write $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathcal{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in $\mathcal{U}$.

Recently, Jung et al. [4] have introduced the following one-parameter families of integral operators:

$$P_\alpha f = P_\alpha f(z) = \frac{2^\alpha}{z \Gamma(\alpha)} \int_0^z \left( \log \frac{z}{t} \right)^{\alpha-1} f(t) \, dt \quad (\alpha > 0)$$  \hspace{1cm} (1.2)

$$Q_\alpha f = Q_\alpha f(z) = \left( \frac{\alpha + \beta}{\beta} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) \, dt \quad (\alpha > 0, \beta > -1)$$  \hspace{1cm} (1.3)

and

$$J_\alpha f = J_\alpha f(z) = \frac{\alpha + 1}{z^\alpha} \int_0^z t^{\alpha-1} f(t) \, dt \quad (\alpha > -1).$$  \hspace{1cm} (1.4)

For $\alpha \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the operators $P_\alpha$, $Q_1$ and $J_\alpha$ were considered by Bernardi ([2], [3]). Further, for real number $\alpha > -1$, the operator $J_\alpha$ was used by Owa and Srivastava [6], and by Srivastava and Owa ([7], [8]).

2000 Mathematics Subject Classification. 30C45.

Key words and phrases. Analytic functions, linear operator, differential subordination.
For \( f(z) \in \mathcal{A} \) Jung et al.\cite{4} have shown that
\[
P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right)^{\alpha} a_n z^n \quad (\alpha > 0),
\]
\[
Q^\alpha_\beta f(z) = z + \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n)}{\Gamma(\alpha + \beta + n)} a_n z^n \quad (\alpha > 0, \beta > -1)
\]
and
\[
J_\alpha f(z) = z + \sum_{n=2}^{\infty} \left( \frac{\alpha + 1}{\alpha + n} \right) a_n z^n \quad (\alpha > -1).
\]

By virtue of (1.5) and (1.7), we see that
\[
J_\alpha f(z) = Q^1_\alpha f(z) \quad (\alpha > -1).
\]

In this paper we derive new subordinations results of analytic functions defined by the above linear operators \( P^\alpha \) and \( Q^\alpha_\beta \).

In order to prove our main results, we recall the following lemmas:

**Lemma 1.1** \((\text{[5]}\)) If \( p(z) = 1 + p_1(z) + p_2(z) + \ldots \) is analytic in \( \mathbb{U} \) and \( h(z) \) is a convex function in \( \mathbb{U} \) with \( h(0) = 1 \) and \( \gamma \) is a complex constant such that \( \text{Re}(\gamma) > 0 \), then
\[
p(z) + \frac{zp'(z)}{\gamma} < h(z)
\]
implies
\[
p(z) < \gamma z^{-\gamma} \int_0^{z} t^{\gamma-1} h(t) \, dt = q(z) < h(z)
\]
and \( q(z) \) is the best dominant.

**Lemma 1.2** \((\text{[1]}\)) For \( a, b, c \) real numbers other than \( 0, -1, -2, \ldots \), and \( c > b > 0 \), we have
\[
\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; z)
\]
\[
F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; \frac{z}{z-1})
\]
\[
F(1, 1; 2; z) = -z^{-1} \ln(1-z)
\]
\[
c(c-1)(z-1) F(a, b; c-1; z) + c[c - (2c - a - b - 1) z] F(a, b; c; z) + (c - a)(c - b) z F(a, b; c+1; z) = 0.
\]

From the identities (1.13) and (1.14), we easily prove the following

**Lemma 1.3.** For any real number \( \zeta \neq 0 \), we have
\[
F(1, 1; 2; \frac{\zeta z}{\zeta z + 1}) = \frac{(1 + \zeta z) \ln(1 + \zeta z)}{\zeta z}
\]
2. Main results

Theorem 2.1. Let $\lambda > 0$, $\alpha > 2$ and suppose that

$$\frac{p^\alpha f(z)}{p^\alpha f(z)} \left[ 1 + \lambda \left( \frac{p^{\alpha-2} f(z) - p^{\alpha-1} f(z)}{p^\alpha f(z)} \right) \right] < \frac{1 + Az}{1 + Bz}$$

Then we have

$$\frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U})$$

where

$$q(z) = (1 + Bz)^{-1} \left[ F(1,1;1+\frac{2}{\lambda} Bz + 1) + \frac{2A}{2+\lambda} F\left(1,1;2+\frac{2}{\lambda} Bz + 1\right) \right]$$

and $q(z)$ is the best dominant. Furthermore,

$$\text{Re}\left\{ \frac{p^\alpha f(z)}{p^\alpha f(z)} \right\} > \rho$$

where

$$\rho = (1 - B)^{-1} \left[ F\left(1,1;1+\frac{2}{\lambda} B + 1\right) - \frac{2A}{2+\lambda} F\left(1,1;2+\frac{2}{\lambda} B + 1\right) \right]$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{p^\alpha f(z)}{p^\alpha f(z)}.$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$. A computation shows that

$$\frac{zp'(z)}{p(z)} = \frac{z(P^{\alpha-1} f(z))'}{p^{\alpha-1} f(z)} - \frac{z(P^\alpha f(z))'}{p^\alpha f(z)}.$$  \hfill (2.5)

By making use of the familiar identity

$$z(P^\alpha f(z))' = 2P^{\alpha-1} f(z) - P^\alpha f(z) \quad (\alpha > 1)$$

in (2.5), we get

$$\frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} - \frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} = \frac{zp'(z)}{2p(z)}.$$  \hfill (2.7)
By using (2.4) and (2.7), we obtain

\[ \frac{P^{\alpha-1} f(z)}{P^\alpha f(z)} \left( 1 + \lambda \left( \frac{P^{\alpha-2} f(z) - P^{\alpha-1} f(z)}{P^\alpha f(z)} \right) \right) = p(z) + \frac{\lambda}{2} z p'(z). \]

Thus, by using Lemma 1.1 for \( \gamma = \frac{2}{\lambda} \), we have

\[ \frac{P^{\alpha-1} f(z)}{P^\alpha f(z)} \prec \left( \frac{2}{\lambda} \right) z^{-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda} - 1} (1 + At) dt = q(z). \]

Now using the identities (1.11) and (1.12), we can rewritten the function \( q(z) \) as

\[ q(z) = \left( \frac{2}{\lambda} \right) \int_0^1 s^{\frac{1}{\lambda} - 1} \frac{1 + Asz}{1 + Bs \bar{z}} ds \]

\[ = \left( \frac{2}{\lambda} \right) \int_0^1 s^{\frac{1}{\lambda} - 1} (1 + Bs \bar{z})^{-1} ds + A \left( \frac{2}{\lambda} \right) z \int_0^1 s^{\frac{1}{\lambda} - 1} ds \]

\[ = (1 + Bz)^{-1} \left[ F\left( 1, 1; 1 + \frac{2}{\lambda}; \frac{B \bar{z}}{Bz + 1} \right) + \frac{2Az}{2 + \lambda} F\left( 1, 1; 2 + \frac{2}{\lambda}; \frac{B \bar{z}}{Bz + 1} \right) \right]. \]

This completes the proof of (2.1).

Next to prove (2.3), it suffices to show that

\[ \inf_{|z| < 1} \{ q(z) \} = q(-1). \quad (2.8) \]

Since for \(-1 \leq B < A \leq 1\), \((1 + Az)/(1 + Bz)\) is convex (univalent) in \( U \), we have for \(|z| \leq r < 1\),

\[ \Re \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}. \quad (2.9) \]

Upon setting

\[ g(s, z) = \frac{1 + Asz}{1 + Bs \bar{z}}, \quad (0 \leq s \leq 1, \ z \in U) \]

and

\[ d\nu(s) = s^{\frac{1}{\lambda} - 1} \left( \frac{2}{\lambda} \right) ds \]

which is a positive measure on \([0, 1]\), we get

\[ q(z) = \int_0^1 g(s, z) d\nu(s) \]

so that

\[ \Re(q(z)) \geq \int_0^1 \left( \frac{1 - Asr}{1 - Bsr} \right) d\nu(s) = q(-r), \quad (|z| \leq r < 1). \]

Letting \( r \to 1^- \) in the above inequality, we obtain the assertion (2.8). Hence the Theorem. \( \square \)

Letting \( \lambda = 2 \) in Theorem 2.1 and using the identities (1.15) and (1.16), we have
Corollary 2.2. Let $\alpha > 2$ and suppose that
\[
\frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \left( 1 + 2 \left( \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} - \frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \right) \right) < \frac{1 + Az}{1 + Bz}
\]  
(2.10)
then we have
\[
\frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} < q(z) < \frac{1 + Az}{1 + Bz} 
\]  
(2.11)
where
\[
q(z) = \begin{cases} 
\frac{A}{B} + (1 - \frac{A}{B}) \ln(1 + Bz), & B \neq 0 \\
1 + \frac{A}{2} z, & B = 0
\end{cases}
\]  
(2.12)
and $q(z)$ is the best dominant. Furthermore,
\[
\text{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \right\} > \rho,
\]
where
\[
\rho = \begin{cases} 
\frac{A}{B} - (1 - \frac{A}{B}) \ln(1 - B), & B \neq 0 \\
1 - \frac{A}{2}, & B = 0
\end{cases}
\]  
(2.13)
Letting $\lambda = 1$ in Theorem 2.1 and using the identities (1.16) and (1.17), we have

Corollary 2.3. Let $\alpha > 2$ and suppose that
\[
\frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \left( 1 + 2 \left( \frac{p^{\alpha-2} f(z)}{p^{\alpha-1} f(z)} - \frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \right) \right) < \frac{1 + Az}{1 + Bz}
\]  
(2.14)
then we have
\[
\frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} < q(z) < \frac{1 + Az}{1 + Bz} 
\]  
(2.15)
where
\[
q(z) = \begin{cases} 
\frac{A}{B} - \frac{2}{B^2} (1 - \frac{A}{B}) \left[ \ln(1 + Bz) - Bz \right], & B \neq 0 \\
1 + \frac{2A}{3} z, & B = 0
\end{cases}
\]  
(2.16)
and $q(z)$ is the best dominant. Furthermore,
\[
\text{Re} \left\{ \frac{p^{\alpha-1} f(z)}{p^\alpha f(z)} \right\} > \rho,
\]
where
\[
\rho = \begin{cases} 
\frac{A}{B} - \frac{2}{B^2} (1 - \frac{A}{B}) \left[ \ln(1 - B) + B \right], & B \neq 0 \\
1 - \frac{2A}{3}, & B = 0
\end{cases}
\]  
(2.17)
Letting $\lambda = 2/3$ in Theorem 2.1 and using the identities (1.17) and (1.18), we have
Corollary 2.4. Let $\alpha > 2$ and suppose that
\[
\frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{\alpha - 2} f(z)}{p^{\alpha - 1} f(z)} - \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right) \right) < \frac{1 + Az}{1 + Bz}
\] (2.18)
then we have
\[
\frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} < q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U})
\] (2.19)
where
\[
q(z) = \begin{cases} 
\frac{4}{B} + \frac{2}{Bz} \left( 1 - \frac{4}{B} \right) \left[ \ln(1 + Bz) - Bz + \frac{(Bz)^2}{2} \right], & B \neq 0 \\
1 + \frac{4A}{3} z, & B = 0
\end{cases}
\] (2.20)
and $q(z)$ is the best dominant. Furthermore,
\[
\Re \left\{ \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right\} > \rho,
\]
where
\[
\rho = \begin{cases} 
\frac{4}{B} - \frac{3}{B^2} \left( 1 - \frac{4}{B} \right) \left[ \ln(1 - B) + B - \frac{B^2}{2} \right], & B \neq 0 \\
1 - \frac{3A}{4}, & B = 0
\end{cases}
\] (2.21)
Letting $B \neq 0$ in Corollaries 2.2, 2.3 and 2.4, respectively, we obtain the following:

Corollary 2.5. Let $\alpha > 2$, then we have the following:

(i) If $\frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{\alpha - 2} f(z)}{p^{\alpha - 1} f(z)} - \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right) \right) < \frac{1 + B \ln(1 - B)}{B + \ln(1 - B)} \frac{z}{1 + Bz}$
\[
\Rightarrow \Re \left\{ \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(ii) If $\frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{\alpha - 2} f(z)}{p^{\alpha - 1} f(z)} - \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right) \right) < \frac{1 + 2B[B + \ln(1 - B)]}{2B[B + \ln(1 - B)] + B^2} \frac{z}{1 + Bz}$
\[
\Rightarrow \Re \left\{ \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(iii) If $\frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{\alpha - 2} f(z)}{p^{\alpha - 1} f(z)} - \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right) \right) < \frac{1 + 3B[\ln(1 - B) + B - (B^2/2)]}{B^3 + 3[\ln(1 - B) + B - (B^2/2)]} \frac{z}{1 + Bz}$
\[
\Rightarrow \Re \left\{ \frac{p^{\alpha - 1} f(z)}{p^{\alpha} f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]
Letting $B = -1$ in Corollary 2.5, we have

Corollary 2.6. Let $\alpha > 2$, then we have the following:
(i) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + 2 \left( \frac{p^{a-2} f(z)}{p^{a-1} f(z)} - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right) \right\} > 2 \ln 2 - 1 \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > 0 \text{ in } \mathbb{C}.
\]

(ii) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + p^{a-2} f(z) - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right\} > 4 \ln 2 - 3 \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > 0 \text{ in } \mathbb{C}.
\]

(iii) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{a-2} f(z)}{p^{a-1} f(z)} - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right) \right\} > 12 \ln 2 - 19 \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > 0 \text{ in } \mathbb{C}.
\]

Letting \( A = 1 - 2\delta, \ 0 \leq \delta < 1 \) and \( B = -1 \) in Corollaries 2.2, 2.3 and 2.4, respectively, we have

**Corollary 2.7.** Let \( \alpha > 2 \), then we have the following:

(i) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + 2 \left( \frac{p^{a-2} f(z)}{p^{a-1} f(z)} - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right) \right\} > \delta \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > (2\delta - 1) + 2(1 - \delta) \ln 2.
\]

(ii) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + p^{a-2} f(z) - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right\} > \delta \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > (2\delta - 1) - 4(1 - \delta)(\ln 2 - 1)
\]

(iii) If \( \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \left( 1 + \frac{2}{3} \left( \frac{p^{a-2} f(z)}{p^{a-1} f(z)} - \frac{p^{a-1} f(z)}{p^a f(z)} \right) \right) \right\} > \delta \), then we have
\[
\Rightarrow \text{Re} \left\{ \frac{p^{a-1} f(z)}{p^a f(z)} \right\} > (2\delta - 1) + 3(1 - \delta)(2\ln 2 - 3)
\]

**Theorem 2.8.** Let \( \alpha > 2, \beta > -1, \lambda > 0 \) and suppose that
\[
\frac{Q_{\beta}^{a-1} f(z)}{Q_{\beta}^a f(z)} \left( 1 + \lambda \left( \frac{Q_{\beta}^{a-2} f(z)}{Q_{\beta}^{a-1} f(z)} - \frac{\alpha + \beta}{\alpha + \beta - 1} \frac{Q_{\beta}^{a-1} f(z)}{Q_{\beta}^a f(z)} + \frac{1}{\alpha + \beta - 1} \right) \right) < \frac{1 + Az}{1 + Bz} \tag{2.22}
\]
then we have
\[
\frac{Q_{\beta}^{a-1} f(z)}{Q_{\beta}^a f(z)} < q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{C}) \tag{2.23}
\]

where
\[
q(z) = (1 + Bz)^{-1} \left[ F \left( 1, 1; 1 + \frac{\alpha + \beta - 1}{\lambda}, \frac{Bz}{Bz + 1} \right) + \frac{(\alpha + \beta - 1)Az}{\alpha + \beta - 1 + \lambda} F \left( 1, 1; 2 + \frac{\alpha + \beta - 1}{\lambda}, \frac{Bz}{Bz + 1} \right) \right]
\]

NEW PROPERTIES OF THE JUNG-KIM-SRIVASTAVA INTEGRAL OPERATORS 211
and \( q(z) \) is the best dominant. Furthermore,

\[
\text{Re} \left\{ \frac{Q_\beta^{a-1} f(z)}{Q_\beta^a f(z)} \right\} > \rho,
\tag{2.24}
\]

where

\[
\rho = (1 - B)^{-1} \left[ F \left( 1, 1; 1 + \frac{\alpha + \beta - 1}{\lambda} \frac{B}{B - 1} + \frac{(\alpha + \beta - 1)A}{\alpha + \beta - 1 + \lambda} \right) \right].
\]

**Proof.** Define the function \( p(z) \) by

\[
p(z) := \frac{Q_\beta^{a-1} f(z)}{Q_\beta^a f(z)}.
\tag{2.25}
\]

Then \( p(z) = 1 + b_1 z + b_2 z + \cdots \) is analytic in \( \mathcal{U} \) with \( p(0) = 1 \). Also, by a simple computation and by making use of the familiar identity

\[
z(Q_\beta^{a-1} f(z))' = (\alpha + \beta)Q_\beta^{a-1} f(z) - (\alpha + \beta - 1)Q_\beta^a f(z) \quad (\alpha > 1, \beta > -1)
\tag{2.26}
\]

we find from (2.25) that

\[
\lambda \left( \frac{Q_\beta^{a-2} f(z)}{Q_\beta^{a-1} f(z)} - \frac{\alpha + \beta}{\alpha + \beta - 1} \frac{Q_\beta^{a-1} f(z)}{Q_\beta^a f(z)} + \frac{1}{\alpha + \beta - 1} \right) = \left( \frac{\lambda}{\alpha + \beta - 1} \right) \frac{zp'(z)}{p(z)}
\tag{2.27}
\]

by using (2.25) and (2.27), we get

\[
\frac{Q_\beta^{a-1} f(z)}{Q_\beta^a f(z)} \left( 1 + \frac{\lambda}{\alpha + \beta - 1} \frac{Q_\beta^{a-2} f(z)}{Q_\beta^{a-1} f(z)} - \frac{\alpha + \beta}{\alpha + \beta - 1} \frac{Q_\beta^{a-1} f(z)}{Q_\beta^a f(z)} + \frac{1}{\alpha + \beta - 1} \right) = p(z) + \left( \frac{\lambda}{\alpha + \beta - 1} \right) zp'(z).
\tag{2.28}
\]

Using Lemma 1.1 for \( \gamma = \frac{\alpha + \beta - 1}{\lambda} \), the estimates (2.23) and (2.24) can be proved on the same lines as that of (2.1) and (2.3). Hence the theorem. \( \square \)

Letting \( \lambda = 1, \alpha = 2 - \beta \) in Theorem 2.8 and using the identities (1.15) and (1.16), we have

**Corollary 2.9.** Let \(-1 < \beta < 0\) and suppose that

\[
\frac{Q_\beta^{1-\beta} f(z)}{Q_\beta^{2-\beta} f(z)} \left( 2 + \frac{Q_\beta^{-\beta} f(z)}{Q_\beta^{1-\beta} f(z)} - 2 \frac{Q_\beta^{1-\beta} f(z)}{Q_\beta^{2-\beta} f(z)} \right) < \frac{1 + A z}{1 + B z}
\tag{2.29}
\]

then we have

\[
\frac{Q_\beta^{1-\beta} f(z)}{Q_\beta^{2-\beta} f(z)} < q(z) < \frac{1 + A z}{1 + B z} \quad (z \in \mathcal{U})
\tag{2.30}
\]
where \( q(z) \) as given in (2.12) and \( q(z) \) is the best dominant. Furthermore,

\[
\text{Re} \left\{ \frac{Q_2^{-\beta} f(z)}{Q_3^{-\beta} f(z)} \right\} > \rho,
\]

where \( \rho \) as given in (2.13).

Letting \( \lambda = 1, \alpha = 3 - \beta \) in Theorem 2.8 and using the identities (1.16) and (1.17), we have

**Corollary 2.10.** Let \(-1 < \beta < 1\) and suppose that

\[
\frac{Q_2^{-\beta} f(z)}{Q_3^{-\beta} f(z)} \left( \frac{1}{3} + \frac{Q_2^{-\beta} f(z)}{Q_3^{-\beta} f(z)} - \frac{1}{3} \right) \leq \frac{1 + Az}{1 + Bz}
\]

then we have

\[
\frac{Q_2^{-\beta} f(z)}{Q_3^{-\beta} f(z)} < q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})
\]

where \( q(z) \) as given in (2.16) and \( q(z) \) is the best dominant. Furthermore,

\[
\text{Re} \left\{ \frac{Q_2^{-\beta} f(z)}{Q_3^{-\beta} f(z)} \right\} > \rho,
\]

where \( \rho \) as given in (2.17).

Letting \( \lambda = 1, \alpha = 4 - \beta \) in Theorem 2.8 and using the identities (1.17) and (1.18), we have

**Corollary 2.11.** Let \(-1 < \beta < 2\) and suppose that

\[
\frac{Q_3^{-\beta} f(z)}{Q_4^{-\beta} f(z)} \left( \frac{1}{3} + \frac{Q_3^{-\beta} f(z)}{Q_4^{-\beta} f(z)} - \frac{1}{3} \right) \leq \frac{1 + Az}{1 + Bz}
\]

then we have

\[
\frac{Q_3^{-\beta} f(z)}{Q_4^{-\beta} f(z)} < q(z) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})
\]

where \( q(z) \) as given in (2.20) and \( q(z) \) is the best dominant. Furthermore,

\[
\text{Re} \left\{ \frac{Q_3^{-\beta} f(z)}{Q_4^{-\beta} f(z)} \right\} > \rho,
\]

where \( \rho \) as given in (2.21).
Letting $B \neq 0$ in Corollaries 2.9, 2.10 and 2.11, respectively, we obtain the following:

**Corollary 2.12.**

(i) Let $-1 < \beta < 0$. If
\[
\frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \left( 2 + \frac{Q^{-\beta}_\beta f(z)}{Q^{1-\beta}_\beta f(z)} - 2 \frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \right)
\]
\[
< \frac{B \ln(1-B) z}{B + \ln(1-B) + B z} \Rightarrow \Re \left\{ \frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(ii) Let $-1 < \beta < 1$. If
\[
\frac{2 - B^2}{2(B + \ln(1-B))} \leq \frac{2B[2B + \ln(1-B)]}{1 + Bz}
\]
\[
\Rightarrow \Re \left\{ \frac{Q^{2-\beta}_\beta f(z)}{Q^{3-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(iii) Let $-1 < \beta < 2$. If
\[
\frac{3 - B^2}{3(B + \ln(1-B)) + B - (B^2/2)}
\]
\[
\Rightarrow \Re \left\{ \frac{Q^{3-\beta}_\beta f(z)}{Q^{4-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

Letting $B = -1$ in Corollary 2.12, we have

**Corollary 2.13.**

(i) Let $-1 < \beta < 0$. If
\[
\Re \left\{ \frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \left( 2 + \frac{Q^{-\beta}_\beta f(z)}{Q^{1-\beta}_\beta f(z)} - 2 \frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \right) \right\}
\]
\[
> \frac{2 \ln 2 - 1}{2 \ln 2 - 2} \approx -0.61 \Rightarrow \Re \left\{ \frac{Q^{1-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(ii) Let $-1 < \beta < 1$. If
\[
\Re \left\{ \frac{Q^{2-\beta}_\beta f(z)}{Q^{3-\beta}_\beta f(z)} \left( 2 + \frac{Q^{-\beta}_\beta f(z)}{Q^{2-\beta}_\beta f(z)} - 2 \frac{Q^{2-\beta}_\beta f(z)}{Q^{3-\beta}_\beta f(z)} \right) \right\}
\]
\[
> \frac{4 \ln 2 - 3}{4 \ln 2 - 2} \approx -0.29 \Rightarrow \Re \left\{ \frac{Q^{2-\beta}_\beta f(z)}{Q^{3-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]

(iii) Let $-1 < \beta < 2$. If
\[
\Re \left\{ \frac{Q^{3-\beta}_\beta f(z)}{Q^{4-\beta}_\beta f(z)} \left( 2 + \frac{Q^{-\beta}_\beta f(z)}{Q^{3-\beta}_\beta f(z)} - 2 \frac{Q^{3-\beta}_\beta f(z)}{Q^{4-\beta}_\beta f(z)} \right) \right\}
\]
\[
> \frac{12 \ln 2 - 19}{12 \ln 2 - 20} \approx 0.91 \Rightarrow \Re \left\{ \frac{Q^{3-\beta}_\beta f(z)}{Q^{4-\beta}_\beta f(z)} \right\} > 0 \text{ in } \mathcal{U}.
\]
NEW PROPERTIES OF THE JUNG-KIM-SRIVASTAVA INTEGRAL OPERATORS

Letting $A = 1 - 2\delta$, $0 \leq \delta < 1$ and $B = -1$ in Corollaries 2.9, 2.10 and 2.11, respectively, we have

**Corollary 2.14.** (i) Let $-1 < \beta < 0$. If
\[ \text{Re} \left\{ Q_{\beta}^{1-\beta} f(z) \left( 2 + \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} - 2 \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right) \right\} > \delta \]
\[ \Rightarrow \text{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right\} > (2\delta - 1) + 2(1-\delta) \ln 2. \]

(ii) Let $-1 < \beta < 1$. If
\[ \text{Re} \left\{ Q_{\beta}^{1-\beta} f(z) \left( \frac{1}{2} + \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} - \frac{3}{2} \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right) \right\} > \delta \]
\[ \Rightarrow \text{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right\} > (2\delta - 1) - 4(1-\delta)(\ln 2 - 1). \]

(iii) Let $-1 < \beta < 2$. If
\[ \text{Re} \left\{ Q_{\beta}^{1-\beta} f(z) \left( \frac{1}{2} + \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} - \frac{3}{2} \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right) \right\} > \delta \]
\[ \Rightarrow \text{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right\} > (2\delta - 1) + 3(1-\delta)(2 \ln 2 - 3). \]

**Acknowledgements**

The author would like to thank the referee for his helpful comments and suggestions.

**References**


Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq, Jordan.

E-mail: bafrasin@yahoo.com