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NEW PROPERTIES OF THE JUNG-KIM-SRIVASTAVA INTEGRAL OPERATORS

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Abstract. The object of the present paper is to prove new subordinations results of analytic functions defined by two integral operators P^α and Q_β^α. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$. For the functions f and g in $\mathscr{A}$, we say that f is subordinate to g in $\mathscr{U}$, and write $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathscr{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = g(w(z))$ in $\mathscr{U}$.

Recently, Jung *et al.* [4] have introduced the following one-parameter families of integral operators:

$$P^\alpha f = P^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\,dt \quad (\alpha > 0) \tag{1.2}$$

$$Q_\beta^\alpha f = Q_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt \quad (\alpha > 0, \beta > -1) \tag{1.3}$$

and

$$J_\alpha f = J_\alpha f(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)\,dt \quad (\alpha > -1). \tag{1.4}$$

For $\alpha \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the operators P^α, Q_1^α and J_α were considered by Bernardi ([2], [3]). Further, for real number $\alpha > -1$, the operator J_α was used by Owa and Srivastava [6], and by Srivastava and Owa ([7], [8]).

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For $f(z) \in \mathcal{A}$ Jung *et al.*[4] have shown that

$$P^{\alpha} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\alpha} a_n z^n \qquad (\alpha > 0), \tag{1.5}$$

$$Q^{\alpha}_{\beta} f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \quad (\alpha > 0, \beta > -1) \tag{1.6}$$

and

$$J_{\alpha} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \qquad (\alpha > -1). \tag{1.7}$$

By virtue of (1.5) and (1.7), we see that

$$J_{\alpha} f(z) = Q^{1}_{\alpha} f(z) \quad (\alpha > -1). \tag{1.8}$$

In this paper is we derive new subordinations results of analytic functions defined by the above linear operators P^{α} and Q^{α}_{β}.

In order to prove our main results, we recall the following lemmas:

Lemma 1.1 ([5])**.** *If* $p(z) = 1 + p_1(z) + p_2(z) + \ldots$ *is analytic in* $\mathcal{U}$ *and* $h(z)$ *is a convex function in* $\mathcal{U}$ *with* $h(0) = 1$ *and* γ *is a complex constant such that* $Re(\gamma) > 0$, *then*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \tag{1.9}$$

implies

$$p(z) \prec \gamma z^{-\gamma} \int_o^z t^{\gamma-1} h(t)\,dt = q(z) \prec h(z) \tag{1.10}$$

and $q(z)$ *is the best dominant.*

Lemma 1.2 ([1])**.** *For* a, b, c *real numbers other than* $0, -1, -2, \ldots$, *and* $c > b > 0$, *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;z) \tag{1.11}$$

$$F(a,b;c;z) = (1-z)^{-a} F(a, c-b; c; \frac{z}{z-1}) \tag{1.12}$$

$$F(1,1;2;z) = -z^{-1}\ln(1-z) \tag{1.13}$$

$$\begin{aligned} &c(c-1)(z-1)F(a,b;c-1;z) + c[c-1-(2c-a-b-1)z]F(a,b;c;z) \\ &\quad + (c-a)(c-b)zF(a,b;c+1;z) = 0. \end{aligned} \tag{1.14}$$

From the identities (1.13) and (1.14), we easily prove the following

Lemma 1.3. *For any real number* $\zeta \neq 0$, *we have*

$$F(1,1;2;\frac{\zeta z}{\zeta z+1}) = \frac{(1+\zeta z)\ln(1+\zeta z)}{\zeta z} \tag{1.15}$$

$$F(1,1;3;\frac{\zeta z}{\zeta z+1}) = \frac{2(1+\zeta z)}{\zeta z}\left[1-\frac{\ln(1+\zeta z)}{\zeta z}\right] \tag{1.16}$$

$$F(1,1;4;\frac{\zeta z}{\zeta z+1}) = \frac{3(1+\zeta z)}{2(\zeta z)^3}\left[2\ln(1+\zeta z)-\zeta z(2-\zeta z)\right] \tag{1.17}$$

$$F(1,1;5;\frac{\zeta z}{\zeta z+1}) = \frac{2(1+\zeta z)}{(\zeta z)^3}\left[\frac{2(\zeta z)^2-3\zeta z+6}{3}-\frac{2\ln(1+\zeta z)}{\zeta z}\right]. \tag{1.18}$$

2. Main results

Theorem 2.1. *Let $\lambda > 0$, $\alpha > 2$ and suppose that*

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\lambda\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right) \prec \frac{1+Az}{1+Bz} \tag{2.1}$$

then we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z\in\mathcal{U}) \tag{2.2}$$

where

$$q(z) = (1+Bz)^{-1}\left[F\left(1,1;1+\frac{2}{\lambda};\frac{Bz}{Bz+1}\right)+\frac{2Az}{2+\lambda}F\left(1,1;2+\frac{2}{\lambda};\frac{Bz}{Bz+1}\right)\right]$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \rho \tag{2.3}$$

where

$$\rho = (1-B)^{-1}\left[F\left(1,1;1+\frac{2}{\lambda};\frac{B}{B-1}\right)-\frac{2A}{2+\lambda}F\left(1,1;2+\frac{2}{\lambda};\frac{B}{B-1}\right)\right]$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}. \tag{2.4}$$

Then $p(z) = 1+b_1z+b_2z+\cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$. A computation shows that

$$\frac{zp'(z)}{p(z)} = \left(\frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)}-\frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)}\right). \tag{2.5}$$

By making use of the familiar identity

$$z(P^{\alpha}f(z))' = 2P^{\alpha-1}f(z)-P^{\alpha}f(z) \quad (\alpha>1) \tag{2.6}$$

in (2.5) , we get

$$\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \frac{zp'(z)}{2p(z)}. \tag{2.7}$$

By using (2.4) and (2.7) , we obtain

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\lambda\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)=p(z)+\frac{\lambda}{2}zp'(z).$$

Thus, by using Lemma 1.1 for $\gamma=\frac{2}{\lambda}$, we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\prec\left(\frac{2}{\lambda}\right)z^{-\left(\frac{2}{\lambda}\right)}\int_0^z\frac{t^{\frac{2}{\lambda}-1}(1+At)dt}{1+Bt}=q(z).$$

Now using the identities (1.11) and (1.12), we can rewritten the function $q(z)$ as

$$\begin{aligned}q(z)&=\left(\frac{2}{\lambda}\right)\int_0^1\frac{s^{\frac{2}{\lambda}-1}(1+Asz)}{1+Bsz}ds\\&=\left(\frac{2}{\lambda}\right)\int_0^1 s^{\frac{2}{\lambda}-1}(1+Bsz)^{-1}ds+A\left(\frac{2}{\lambda}\right)z\int_0^1 s^{\frac{2}{\lambda}}(1+Bsz)^{-1}ds\\&=(1+Bz)^{-1}\left[F\left(1,1;1+\frac{2}{\lambda};\frac{Bz}{Bz+1}\right)+\frac{2Az}{2+\lambda}F\left(1,1;2+\frac{2}{\lambda};\frac{Bz}{Bz+1}\right)\right].\end{aligned}$$

This completes the proof of (2.1).

Next to prove (2.3), it suffices to show that

$$\inf_{|z|<1}\{q(z)\}=q(-1).\tag{2.8}$$

Since for $-1\le B<A\le 1$, $(1+Az)/(1+Bz)$ is convex (univalent) in $\mathcal{U}$, we have for $|z|\le r<1$,

$$\operatorname{Re}\left(\frac{1+Az}{1+Bz}\right)\ge\frac{1-Ar}{1-Br}.\tag{2.9}$$

Upon setting

$$g(s,z)=\frac{1+Asz}{1+Bsz},\qquad(0\le s\le 1,\ z\in\mathcal{U})$$

and

$$d\nu(s)=s^{\left(\frac{2}{\lambda}-1\right)}\left(\frac{2}{\lambda}\right)ds$$

which is a positive measure on $[0,1]$, we get

$$q(z)=\int_0^1 g(s,z)d\nu(s)$$

so that

$$\operatorname{Re}\{q(z)\}\ge\int_0^1\left(\frac{1-Asr}{1-Bsr}\right)d\nu(s)=q(-r),\qquad(|z|\le r<1).$$

Letting $r\to 1^-$ in the above inequality, we obtain the assertion (2.8). Hence the Theorem. □

Letting $\lambda=2$ in Theorem 2.1 and using the identities (1.15) and (1.16), we have

Corollary 2.2. *Let $\alpha > 2$ and suppose that*

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+2\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right) \prec \frac{1+Az}{1+Bz} \tag{2.10}$$

then we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.11}$$

where

$$q(z) = \begin{cases} \frac{A}{B}+(1-\frac{A}{B})\frac{\ln(1+Bz)}{Bz}, & B \neq 0 \\ 1+\frac{A}{2}z, & B = 0 \end{cases} \tag{2.12}$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \rho,$$

where

$$\rho = \begin{cases} \frac{A}{B}-(1-\frac{A}{B})\frac{\ln(1-B)}{B}, & B \neq 0 \\ 1-\frac{A}{2}, & B = 0. \end{cases}. \tag{2.13}$$

Letting $\lambda = 1$ in Theorem 2.1 and using the identities (1.16) and (1.17), we have

Corollary 2.3. *Let $\alpha > 2$ and suppose that*

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right) \prec \frac{1+Az}{1+Bz} \tag{2.14}$$

then we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.15}$$

where

$$q(z) = \begin{cases} \frac{A}{B}-\frac{2}{B^2}(1-\frac{A}{B})\left[\frac{\ln(1+Bz)-Bz}{z^2}\right], & B \neq 0 \\ 1+\frac{2A}{3}z, & B = 0 \end{cases} \tag{2.16}$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \rho,$$

where

$$\rho = \begin{cases} \frac{A}{B}-\frac{2}{B^2}(1-\frac{A}{B})[\ln(1-B)+B], & B \neq 0 \\ 1-\frac{2A}{3}, & B = 0 \end{cases}. \tag{2.17}$$

Letting $\lambda = 2/3$ in Theorem 2.1 and using the identities (1.17) and (1.18), we have

Corollary 2.4. *Let $\alpha > 2$ and suppose that*

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{2}{3}\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\prec\frac{1+Az}{1+Bz} \tag{2.18}$$

then we have

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\prec q(z)\prec\frac{1+Az}{1+Bz} \qquad (z\in\mathcal{U}) \tag{2.19}$$

where

$$q(z)=\begin{cases}\frac{A}{B}+\frac{3}{(Bz)^3}(1-\frac{A}{B})\left[\ln(1+Bz)-Bz+\frac{(Bz)^2}{2}\right], & B\neq 0\\ 1+\frac{3A}{4}z, & B=0\end{cases} \tag{2.20}$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\}>\rho,$$

where

$$\rho=\begin{cases}\frac{A}{B}-\frac{3}{B^3}(1-\frac{A}{B})\left[\ln(1-B)+B-\frac{B^2}{2}\right], & B\neq 0\\ 1-\frac{3A}{4}, & B=0\end{cases} \tag{2.21}$$

Letting $B\neq 0$ in Corollaries 2.2, 2.3 and 2.4, respectively, we obtain the following :

Corollary 2.5. *Let $\alpha > 2$, then we have the following:*

(i) *If* $\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+2\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\prec\frac{1+\frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz}$
$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\}>0$ *in* $\mathcal{U}$.

(ii) *If* $\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\prec\frac{1+\frac{2B[B+\ln(1-B)]}{2(B+\ln(1-B))+B^2}z}{1+Bz}$
$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\}>0$ *in* $\mathcal{U}$.

(iii) *If* $\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{2}{3}\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\prec\frac{1+\frac{3B[\ln(1-B)+B-(B^2/2)]}{B^3+3[\ln(1-B)+B-(B^2/2)]}z}{1+Bz}$
$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\}>0$ *in* $\mathcal{U}$.

Letting $B=-1$ in Corollary 2.5, we have

Corollary 2.6. *Let $\alpha > 2$, then we have the following:*

(i) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+2\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\right\} > \frac{2\ln 2-1}{2\ln 2-2} \approx -0.61$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > 0$ in $\mathcal{U}$.

(ii) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right\} > \frac{4\ln 2-3}{4\ln 2-2} \approx -0.29$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > 0$ in $\mathcal{U}$

(iii) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{2}{3}\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\right\} > \frac{12\ln 2-19}{12\ln 2-20} \approx 0.91$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > 0$ in $\mathcal{U}$

Letting $A = 1-2\delta$, $0 \le \delta < 1$ and $B = -1$ in Corollaries2.2, 2.3 and 2.4, respectively, we have

Corollary 2.7. *Let $\alpha > 2$, then we have the following:*

(i) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+2\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\right\} > \delta$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > (2\delta-1)+2(1-\delta)\ln 2.$

(ii) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right\} > \delta$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > (2\delta-1)-4(1-\delta)(\ln 2-1)$

(iii) If $Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\left(1+\frac{2}{3}\left(\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}-\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right)\right)\right\} > \delta$

$\Rightarrow Re\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > (2\delta-1)+3(1-\delta)(2\ln 2-3).$

Theorem 2.8. *Let $\alpha > 2$, $\beta > -1$, $\lambda > 0$ and suppose that*

$$\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}\left(1+\lambda\left(\frac{Q_{\beta}^{\alpha-2}f(z)}{Q_{\beta}^{\alpha-1}f(z)}-\frac{\alpha+\beta}{\alpha+\beta-1}\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)}+\frac{1}{\alpha+\beta-1}\right)\right) \prec \frac{1+Az}{1+Bz} \tag{2.22}$$

then we have

$$\frac{Q_{\beta}^{\alpha-1}f(z)}{Q_{\beta}^{\alpha}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.23}$$

where

$$q(z) = (1+Bz)^{-1}\left[F\left(1,1;1+\frac{\alpha+\beta-1}{\lambda};\frac{Bz}{Bz+1}\right)+\frac{(\alpha+\beta-1)Az}{\alpha+\beta-1+\lambda}F\left(1,1;2+\frac{\alpha+\beta-1}{\lambda};\frac{Bz}{Bz+1}\right)\right]$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \rho, \tag{2.24}$$

where

$$\rho = (1-B)^{-1}\left[F\left(1,1;1+\frac{\alpha+\beta-1}{\lambda};\frac{B}{B-1}\right) - \frac{(\alpha+\beta-1)A}{\alpha+\beta-1+\lambda}F\left(1,1;2+\frac{\alpha+\beta-1}{\lambda};\frac{B}{B-1}\right)\right].$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}. \tag{2.25}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$. Also, by a simple computation and by making use of the familiar identity

$$z(Q_\beta^{\alpha}f(z))' = (\alpha+\beta)Q_\beta^{\alpha-1}f(z) - (\alpha+\beta-1)Q_\beta^{\alpha}f(z) \quad (\alpha > 1, \beta > -1) \tag{2.26}$$

we find from (2.25) that

$$\lambda\left(\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} - \frac{\alpha+\beta}{\alpha+\beta-1}\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} + \frac{1}{\alpha+\beta-1}\right) = \left(\frac{\lambda}{\alpha+\beta-1}\right)\frac{zp'(z)}{p(z)} \tag{2.27}$$

by using (2.25) and (2.27), we get

$$\begin{aligned}&\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\left(1+\lambda\left(\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} - \frac{\alpha+\beta}{\alpha+\beta-1}\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} + \frac{1}{\alpha+\beta-1}\right)\right)\\ &\quad = p(z) + \left(\frac{\lambda}{\alpha+\beta-1}\right)zp'(z).\end{aligned} \tag{2.28}$$

Using Lemma 1.1 for $\gamma = \frac{\alpha+\beta-1}{\lambda}$, the estimates (2.23) and (2.24) can be proved on the same lines as that of (2.1) and (2.3). Hence the theorem. □

Letting $\lambda = 1$, $\alpha = 2-\beta$ in Theorem 2.8 and using the identities (1.15) and (1.16), we have

Corollary 2.9. *Let $-1 < \beta < 0$ and suppose that*

$$\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\left(2 + \frac{Q_\beta^{-\beta}f(z)}{Q_\beta^{1-\beta}f(z)} - 2\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right) \prec \frac{1+Az}{1+Bz} \tag{2.29}$$

then we have

$$\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.30}$$

where $q(z)$ as given in (2.12) and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{Q_{\beta}^{1-\beta}f(z)}{Q_{\beta}^{2-\beta}f(z)}\right\} > \rho,$$

where ρ as given in (2.13).

Letting $\lambda = 1$, $\alpha = 3-\beta$ in Theorem 2.8 and using the identities (1.16) and (1.17), we have

Corollary 2.10. *Let $-1 < \beta < 1$ and suppose that*

$$\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)}\left(\frac{3}{2}+\frac{Q_{\beta}^{1-\beta}f(z)}{Q_{\beta}^{2-\beta}f(z)}-\frac{3}{2}\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)}\right) \prec \frac{1+Az}{1+Bz} \tag{2.31}$$

then we have

$$\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.32}$$

where $q(z)$ as given in (2.16) and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)}\right\} > \rho,$$

where ρ as given in (2.17).

Letting $\lambda = 1$, $\alpha = 4-\beta$ in Theorem 2.8 and using the identities (1.17) and (1.18), we have

Corollary 2.11. *Let $-1 < \beta < 2$ and suppose that*

$$\frac{Q_{\beta}^{3-\beta}f(z)}{Q_{\beta}^{4-\beta}f(z)}\left(\frac{4}{3}+\frac{Q_{\beta}^{2-\beta}f(z)}{Q_{\beta}^{3-\beta}f(z)}-\frac{4}{3}\frac{Q_{\beta}^{3-\beta}f(z)}{Q_{\beta}^{4-\beta}f(z)}\right) \prec \frac{1+Az}{1+Bz} \tag{2.33}$$

then we have

$$\frac{Q_{\beta}^{3-\beta}f(z)}{Q_{\beta}^{4-\beta}f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}) \tag{2.34}$$

where $q(z)$ as given in (2.20) and $q(z)$ is the best dominant. Furthermore,

$$Re\left\{\frac{Q_{\beta}^{3-\beta}f(z)}{Q_{\beta}^{4-\beta}f(z)}\right\} > \rho,$$

where ρ as given in (2.21).

Letting $B \neq 0$ in Corollaries 2.9, 2.10 and 2.11, respectively, we obtain the following :

Corollary 2.12. (i) *Let* $-1<\beta<0$. *If* $\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\left(2+\frac{Q_\beta^{-\beta}f(z)}{Q_\beta^{1-\beta}f(z)}-2\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right)$

$$\prec \frac{1+\frac{B\ln(1-B)}{B+\ln(1-B)}z}{1+Bz} \Rightarrow Re\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

(ii) *Let* $-1<\beta<1$. *If* $\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\left(\frac{3}{2}+\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}-\frac{3}{2}\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right)$

$$\prec \frac{1+\frac{2B[B+\ln(1-B)]}{2(B+\ln(1-B))+B^2}z}{1+Bz} \Rightarrow Re\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

(iii) *Let* $-1<\beta<2$. *If* $\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\left(\frac{4}{3}+\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}-\frac{4}{3}\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right)$

$$\prec \frac{1+\frac{3B[\ln(1-B)+B-(B^2/2)]}{B^3+3[\ln(1-B)+B-(B^2/2)]}z}{1+Bz} \Rightarrow Re\left\{\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

Letting $B=-1$ in Corollary 2.12, we have

Corollary 2.13. (i) *Let* $-1<\beta<0$. *If* $Re\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\left(2+\frac{Q_\beta^{-\beta}f(z)}{Q_\beta^{1-\beta}f(z)}-2\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right)\right\}$

$$> \frac{2\ln 2-1}{2\ln 2-2}\approx -0.61 \Rightarrow Re\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

(ii) *Let* $-1<\beta<1$. *If* $Re\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\left(\frac{3}{2}+\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}-\frac{3}{2}\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right)\right\}$

$$> \frac{4\ln 2-3}{4\ln 2-2}\approx -0.29 \Rightarrow Re\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

(iii) *Let* $-1<\beta<2$. *If* $Re\left\{\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\left(\frac{4}{3}+\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}-\frac{4}{3}\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right)\right\}$

$$> \frac{12\ln 2-19}{12\ln 2-20}\approx 0.91 \Rightarrow Re\left\{\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right\}>0 \ in\ \mathscr{U}.$$

Letting $A = 1-2\delta$, $0 \le \delta < 1$ and $B = -1$ in Corollaries 2.9, 2.10 and 2.11, respectively, we have

Corollary 2.14. (i) *Let* $-1 < \beta < 0$. *If* $Re\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\left(2+\frac{Q_\beta^{-\beta}f(z)}{Q_\beta^{1-\beta}f(z)}-2\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right)\right\} > \delta$

$$\Rightarrow \ Re\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right\} > (2\delta-1)+2(1-\delta)\ln 2.$$

(ii) *Let* $-1 < \beta < 1$. *If* $Re\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\left(\frac{3}{2}+\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}-\frac{3}{2}\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right)\right\} > \delta$

$$\Rightarrow Re\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right\} > (2\delta-1)-4(1-\delta)(\ln 2-1).$$

(iii) *Let* $-1 < \beta < 2$. *If* $Re\left\{\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\left(\frac{4}{3}+\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}-\frac{4}{3}\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right)\right\} > \delta$

$$\Rightarrow \ Re\left\{\frac{Q_\beta^{3-\beta}f(z)}{Q_\beta^{4-\beta}f(z)}\right\} > (2\delta-1)+3(1-\delta)(2\ln 2-3).$$

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