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COEFFICIENT ESTIMATE FOR A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. For reals A, B, C, D such that $-1 \le D \le B < A \le C \le 1$, a subclass $K_s(A,B;C,D)$ of analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disc $E = \{z : |z| < 1\}$ is introduced. The object of the present paper is to determine the coefficient estimate for functions $f(z)$ belonging to the class $K_s(A,B;C,D)$.

1. Introduction

Let U denote the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1, \quad z \in E.$$

Let S be the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

which are regular and univalent in E.

Let S_s^* denote the class of functions $f(z) \in S$ and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E.$$

These functions are called Starlike with respect to symmetric points and were introduced by Sakaguchi [4]. After this Goel and Mehrok [2] introduced by a sub-class $S_s^*(A,B)$ of S_s^*. $S_s^*(A,B)$ be the class of functions $f(z) \in S$ which satisfy the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \le B < A \le 1,\ z \in E.$$

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Let $K_s(A,B;C,D)$ be the class consisting of functions $f(z) \in S$ and satisfying the condition

$$\frac{2zf'(z)}{(g(z)-g(-z))} \prec \frac{1+Cz}{1+Dz}, \quad -1 \le D \le B < A \le C \le 1, \; z \in E$$

where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*(A,B).$$

Obviously $K_s \equiv K_s(1,-1;1,-1)$ and $K_s(A,B) = K_s(A,B;1,-1)$.

By definition of subordination it follows that $f \in K_s(A,B;C,D)$ if and only if

$$\frac{2zf'(z)}{g(z)-g(-z)} = \frac{1+Cw(z)}{1+Dw(z)} = P(z), \quad w(z) \in U \tag{1.3}$$

where

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k. \tag{1.4}$$

We obtain the coefficient estimate for the class $K_s(A,B;C,D)$.

2. Some preliminary lemmas

We shall require the following lemmas.

Lemma 2.1. *If $P(z)$ is given by (1.3), then*

$$|p_n| \le (C-D).$$

This lemma is due to Goel and Mahrok [2].

Lemma 2.2. *Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*(A,B)$, then for $n \ge 1$,*

$$|b_{2n}| \le \frac{(A-B)}{n!2^n} \prod_{j=1}^{n-1} (A-B+2j)$$

and

$$|b_{2n+1}| \le \frac{(A-B)}{n!2^n} \prod_{j=1}^{n-1} (A-B+2j).$$

This result was established by Goel and Mahrok [2].

3. Main result

Theorem 3.1. *Let $f \in K_s(A,B;C,D)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(C-D)}{n!2^n}\prod_{j=1}^{n-1}(A-B+2j) \tag{3.1}$$

$$|a_{2n+1}| \leq \frac{1}{2n+1}\Big\{\Big[(C-D)+\frac{(A-B)}{2n}\Big]\Big[\frac{1}{(n-1)!2^{n-1}}\prod_{j=1}^{n-1}(A-B+2j)\Big]\Big\}. \tag{3.2}$$

Proof. As $g \in S_s^*(A,B)$, it follows that

$$2zg'(z) = (g(z)-g(-z))K(z) \quad \text{for } z \in E \tag{3.3}$$

$$\text{where} \quad K(z) = 1+d_1z+d_2z^2+d_3z^3+\cdots.$$

On equating the coefficients of like powers of z in (3.3), we get

$$2b_2 = d_1, \qquad 2b_3 = d_2, \tag{3.4}$$

$$4b_4 = d_3+b_3d_1, \qquad 4b_5 = d_4+b_3d_2, \tag{3.5}$$

Continuing in this way , we have

$$2nb_{2n} = d_{2n-1}+b_3d_{2n-3}+b_5d_{2n-5}+\cdots+b_{2n-1}d_1, \tag{3.6}$$

$$2nb_{2n+1} = d_{2n}+b_3d_{2n-2}+b_5d_{2n-4}+\cdots+b_{2n-1}d_2. \tag{3.7}$$

From (1.3) and (1.4), we have

$$\begin{aligned} z+2a_2z^2+3a_3z^3+\cdots+2na_{2n}z^{2n}+(2n+1)a_{2n+1}z^{2n+1}+\cdots \\ = (z+b_3z^3+b_5z^5+\cdots+b_{2n-1}z^{2n-1}+b_{2n+1}z^{2n+1}+\cdots) \\ \cdot(1+p_1z+p_2z^2+\cdots+p_{2n}z^{2n}+p_{2n+1}z^{2n+1}+\cdots). \end{aligned}$$

On equating the coefficients, we otain

$$2a_2 = p_1, \qquad 3a_3 = p_2+b_3, \tag{3.8}$$

$$4a_4 = p_3+b_3p_1, \qquad 5a_5 = p_4+b_3p_2+b_5, \tag{3.9}$$

and so on

$$2na_{2n} = p_{2n-1}+b_3p_{2n-3}+b_5p_{2n-5}+\cdots+b_{2n-1}p_1, \tag{3.10}$$

$$(2n+1)a_{2n+1} = p_{2n}+b_3p_{2n-2}+b_5p_{2n-4}+\cdots+b_{2n-1}p_2+b_{2n+1}. \tag{3.11}$$

Using Lemma 2.1 and equation (3.8), we get

$$2|a_2| \leq C-D, \qquad 3|a_3| \leq (C-D)+\frac{(A-B)}{2}.$$

Again applying Lemma 2.1 and using equation (3.4) and (3.5), we obtain from (3.9)

$$4|a_4| \leq \frac{(C-D)(A-B+2)}{2}, \qquad 5|a_5| \leq \frac{(A-B+2)[(A-B)+4(C-D)]}{8}.$$

It follow that (3.1) and (3.2) hold for $n = 1, 2$.

We now prove (3.1) and (3.2) by induction.

(3.10) and (3.11) in conjunction with Lemma 2.1 yield

$$|a_{2n}| \leq \frac{(C-D)}{2n}\Big[1+\sum_{k=1}^{n-1}|b_{2k+1}|\Big]. \tag{3.12}$$

and

$$|a_{2n+1}| \leq \frac{1}{2n+1}\Big\{(C-D)\Big[1+\sum_{k=1}^{n-1}|b_{2k+1}|\Big]+|b_{2n+1}|\Big\}. \tag{3.13}$$

Again by using Lemma 2.1 in (3.7), we have

$$|b_{2n+1}| \leq \frac{(A-B)}{2n}\Big[1+\sum_{k=1}^{n-1}|b_{2k+1}|\Big]. \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$|a_{2n+1}| \leq \frac{1}{2n+1}\Big\{\Big[(C-D)+\frac{(A-B)}{2n}\Big]\Big[1+\sum_{k=1}^{n-1}|b_{2k+1}|\Big]\Big\}. \tag{3.15}$$

We assume that (3.1) and (3.2) holds for $k = 3, 4, \ldots, (n-1)$.

Using Lemma 2.2 in (3.12) and (3.15), we obtain

$$|a_{2n}| \leq \frac{(C-D)}{2n}\Big[1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\Big] \tag{3.16}$$

and

$$\begin{aligned} |a_{2n+1}| \leq \frac{1}{2n+1}\Big\{(C-D)\Big[1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\Big] \\ +\frac{(A-B)}{2n}\Big[1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\Big]\Big\}. \end{aligned} \tag{3.17}$$

In order to prove (3.1) , it is sufficient to show that

$$\frac{(C-D)}{2m}\Big[1+\sum_{k=1}^{m-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\Big] = \frac{(C-D)}{m!2^m}\prod_{j=1}^{m-1}(A-B+2j), \quad (m = 3, 4, \ldots) \tag{3.18}$$

(3.18) is valid for $m = 3$.

Let us suppose that (3.18) is true for all m, $3 < m \leq (n-1)$. Then from (3.16) , we have

$$\frac{(C-D)}{2n}\Big[1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\Big]$$

$$= \frac{(n-1)}{n}\left\{\frac{(C-D)}{2(n-1)}\left[1+\sum_{k=1}^{n-2}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}$$
$$+\frac{(C-D)}{2n}\cdot\frac{(A-B)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)$$
$$= \frac{(n-1)}{n}\cdot\frac{(C-D)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)$$
$$+\frac{(C-D)}{2n}\cdot\frac{(A-B)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)$$
$$= \frac{(C-D)}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\frac{(A-B+2(n-1))}{2n}$$
$$= \frac{(C-D)}{n!2^n}\prod_{j=1}^{n-1}(A-B+2j).$$

Thus (3.18) holds for $m=n$ and hence (3.1) follows.

Now from (3.17), we have

$$|a_{2n+1}| \le \frac{1}{2n+1}\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]\right\}. \tag{3.19}$$

From (3.18) , we have

$$1+\sum_{k=1}^{n-1}\frac{(A-B)}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j) = \frac{1}{(n-1)!2^{n-1}}\prod_{j=1}^{n-1}(A-B+2j) \tag{3.20}$$

From (3.19) and (3.20), we have

$$|a_{2n+1}| \le \frac{1}{2n+1}\left\{\left[(C-D)+\frac{(A-B)}{2n}\right]\left[\frac{1}{(n-1)!2^{n-1}}\prod_{j=1}^{n-1}(A-B+2j)\right]\right\}$$

which proves (3.2).

Putting $A=C=1$ and $B=D=-1$ in the above result , we get the following

Corollary 1. *Let $f(z)$ be schlicht and starlike with respect to symmetric points in the unit disc E, having the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, then*

$$|a_n| \le 1 \text{ for any natural number } n.$$

This result was proved by Das and Singh [1].

For $C=1$ and $D=-1$, we have the following result for the class $K_s(A,B)$.

Corollary 2. *Let $f \in K_s(A,B)$, then for $n \geq 1$,*

$$|a_{2n+1}| \leq \frac{1}{2n+1}\left\{\left[2+\frac{(A-B)}{2n}\right]\left[\frac{1}{(n-1)!2^{n-1}}\prod_{j=1}^{n-1}(A-B+2j)\right]\right\}.$$

Remark. Janteng and Halim [3] proved that, for $f \in K_s(A,B)$ and for $n \geq 1$,

$$|a_{2n+1}| \leq \frac{(A-B)}{(2n+1)(n-1)!2^{n-1}}\prod_{j=1}^{n-1}(A-B+2j).$$

This result is not justified.

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