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WEIGHTED SHARING AND UNIQUENESS OF MERUMORPHIC FUNCTIONS

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Abstract. In this paper, we study with a weighted sharing method the uniqueness problem of $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ sharing one value and obtain some results which extend and improve the results due to Hong-Yan Xu and Ting-Bin Cao.

1. Introduction

Let f be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

$$T(r,f),\ m(r,f),\ N(r,f),\ \overline{N}(r,f),\ldots$$

(See Hayman [3], Yang [6] and Yi and Yang [7]). We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$,

as $r\to+\infty$, possibly outside of a set with finite measure. For any constant $'a'$, we define

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{(f-a)}\right)}{T(r,f)},$$

Let $'a'$ be a finite complex number and k a positive integer. We denote by $N_{k)}\left(r,\frac{1}{(f-a)}\right)$ the counting function for the zeros of $f(z)-a$ with the multiplicity $\le k$, and by $\overline{N}_{k)}\left(r,\frac{1}{(f-a)}\right)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}\left(r,\frac{1}{(f-a)}\right)$ be the counting function for the zeros of $f(z)-a$ with multiplicity atleast k, and $\overline{N}_{(k}\left(r,\frac{1}{(f-a)}\right)$ be the corresponding one for which the multiplicity is not counted. Set

$$N_k\left(r,\frac{1}{(f-a)}\right)=\overline{N}\left(r,\frac{1}{(f-a)}\right)+\overline{N}_{(2}\left(r,\frac{1}{(f-a)}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{(f-a)}\right).$$

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We define

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_k\left(r, \frac{1}{(f-a)}\right)}{T(r,f)}.$$

Let g be a meromorphic function. If $f(z)-a$ and $g(z)-a$, assume the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value $'a'$ CM, where $'a'$ is a complex number. Similarly, we say that f and g share a IM, provided that $f(z)-a$ and $g(z)-a$ have same multiplicities.

In 1996, Fang proved the following result.

Theorem A([1]). *Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k+4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share the value 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.*

In 1997, Yang and Hua obtained a unicity theorem corresponding to above result.

Theorem B([8]). *Let f and g be two nonconstant entire functions, $n \geq 6$ a positive integer. If f^nf' and g^ng' share 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = 1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002, Fang proved the following result.

Theorem C([2]). *Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share the value 1 CM, then $f \equiv g$.*

In 2008, Zhang and Lin, Zhang, Chen and Lin extended Theorem C and obtain the following results.

Theorem D([10]). *Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k+m+4$, and λ, μ be constants such that $|\lambda|+|\mu| \neq 0$. If $[f^n(\mu f^m+\lambda)]^{(k)}$ and $[g^n(\mu g^m+\lambda)]^{(k)}$ share 1 CM, then*

(i) *when $\lambda\mu \neq 0$, $f \equiv g$.*

(ii) *when $\lambda\mu = 0$, either $f \equiv tg$, where t is a constant satisfying $t^{n+m} = 1$, or $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(-1)^k\lambda^2(c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$ or $(-1)^k\mu^2(c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$.*

Theorem E([11]). *Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k+m+4$, and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ or $P(z) \equiv c_0$, where $a_0 \neq 0$, $a_1, \ldots, a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then*

(i) *when* $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, *either* $f \equiv tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, \ldots, m$, *or* f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_1\omega_1 + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_1\omega_2 + a_0)$;

(ii) *when* $P(z) = c_0$, *either* $f(z) = c_1/\sqrt[n]{c_0}e^{cz}$, $g(z) = c_2/\sqrt[n]{c_0}e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, *or* $f = tg$ *for a constant* t *such that* $t^n = 1$.

In 2009, H.-Y. Xu and T.-B. Cao proved the following result.

Theorem F([5]). *Let* f *and* g *be two nonconstant entire functions, and let* n, m *and* k *be three positive integers with* $n \geq 5k+5m+8$. *If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share (1,0), then the conclusion of Theorem E still holds.*

Theorem G([5]). *Let* f *and* g *be two nonconstant entire functions, and let* n, m *and* k *be three positive integers with* $n > \frac{9}{2}m + 4k + \frac{9}{2}$. *If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share (1,1), then the conclusion of Theorem E still holds.*

Theorem H([5]). *Let* f *and* g *be two nonconstant entire functions, and let* n, m *and* k *be three positive integers with* $n \geq 3m+3k+5$. *If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* $(1,2)$, *then the conclusion of Theorem E still holds.*

In this paper, by introducing the notion of multiplicity, we reduce and improve Theorems F, G and H. Also we extend these theorems to meromorphic functions and obtain the following results.

Theorem 1.1. *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast* s, *where* s *is a positive integer. Let* $P(f) = a_mf^m + a_{m-1}f^{m-1} + \cdots + a_1f + a_0$, $(a_m \neq 0)$, *and* $a_i (i = 0, 1, \ldots, m)$ *is the first nonzero coefficient from the right, and let* n, k, m *be three positive integers. If* $[f^nP(f)]^{(k)}$ *and* $[g^nP(g)]^{(k)}$ *share* $(1,l)$ *and one of the following conditions holds:*

(i) $l \geq 2$ *and* $s(n+m) > 3k+10$

(ii) $l = 1$ *and* $s(n+m) > 5k+13$

(iii) $l = 0$ *and* $s(n+m) > 9k+16$

then either $f = tg$ *for a constant* t *such that* $t^d = 1$, *where* $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ *for some* $i = 0, 1, \ldots, m$, *or* f *and* g *satisfy the algebraic equation* $R(f,g) \equiv 0$, *where* $R(\omega_1, \omega_2) = \omega_1^nP(\omega_1) - \omega_2^nP(\omega_2)$.

Theorem 1.2. *Let* f *and* g *be two non-constant entire functions, whose zeros and poles are of multiplicities atleast* s, *where* s *is a positive integer. Let* $P(f) = a_mf^m + a_{m-1}f^{m-1} + \cdots + a_1f + a_0$,

$(a_m \neq 0)$, and $a_i (i = 0, 1, \ldots, m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and one of the following conditions holds:

(i) $l \geq 2$ *and* $s(n+m) > 3k+5$

(ii) $l = 1$ *and* $s(n+m) > 4k+6$

(iii) $l = 0$ *and* $s(n+m) > 5k+8$

then either $f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Remark. In Theorem 1.2, giving specific values for s, we get the following interesting cases:

(i) If $s = 1$, then for $l \geq 2$ we get $n > 3k+5-m$, for $l = 1$ we get $n > 4k+6-m$ and for $l = 0$ we get $n > 5k+8-m$.

(ii) If $s = 2$, then for $l \geq 2$ we get $n > \frac{3k+5}{2} - m$, for $l = 1$ we get $n > 2k+3-m$ and for $l = 0$ we get $n > \frac{5k+8}{2} - m$.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

2. Some Lemmas

Lemma 2.1 ([3])**.** *Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.2 ([9])**.** *Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + \cdots + a_n f^n$, where a_0, $a_1, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r,f) + S(r,f).$$

Lemma 2.3 ([4, 12]). *Let f be a non-constant meromorphic function and k be a positive integer, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)$$
$$\le (p+k)\overline{N}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

This Lemma can be obtained immediately from the proof of Lemma 2.3 *in [4] which is the case* $p=2$.

Lemma 2.4 ([13]). *Let F and G be two nonconstant meromorphic functions. If F and G share 1 IM, then* $\overline{N}_L(r,\frac{1}{F-1})\le\overline{N}(r,\frac{1}{F})+\overline{N}(r,F)+S(r,F)$.

Lemma 2.5 ([5]). *Let f and g be two nonconstant entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share* $(1,l)$ $(l=0,1,2)$. *Then*

(i) *If* $l=0$,
$\Theta(0,f)+\delta_k(0,f)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\delta_{k+2}(0,f)+\delta_{k+2}(0,g)>5$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$;

(ii) *If* $l=1$,
$\frac{1}{2}\left[\Theta(0,f)+\delta_k(0,f)+\delta_{k+2}(0,f)\right]+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\Theta(0,g)+\delta_k(0,g)>\frac{9}{2}$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$;

(iii) *If* $l=2$,
$\Theta(0,f)+\delta_k(0,f)+\delta_{k+1}(0,f)+\delta_{k+2}(0,g)>3$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$.

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions, $k\ (\ge 1)$ and $l\ (\ge 0)$ be integers. If $f^{(k)}$ and $g^{(k)}$ share* $(1,l)$ $(l=0,1,2)$. *Then*

(i) *If* $l\ge 2$,
$(k+2)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>k+7$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$;

(ii) *If* $l=1$,
$(2k+3)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)+\delta_{k+2}(0,f)>2k+9$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$;

(iii) *If* $l=0$,
$(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)>4k+13$, *then either* $f^{(k)}g^{(k)}=1$ *or* $f\equiv g$.

Proof. Let

$$\Phi(z)=\left(\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}\right)-\left(\frac{g^{(k+2)}}{g^{(k+1)}}-2\frac{g^{(k+1)}}{g^{(k)}-1}\right).\tag{2.1}$$

Suppose that $\Phi(z) \neq 0$. If z_0 is a common simple 1-point of $f^{(k)}(z)$ and $f^{(k)}(z)$, substituting their Taylor series at z_0 into (2.1), we can get $\Phi(z_0) = 0$. Thus we have,

$$N_E^{1)}\left(r, \frac{1}{f^{(k)}-1}\right) = N_E^{1)}\left(r, \frac{1}{g^{(k)}-1}\right) \leq \overline{N}\left(r, \frac{1}{\Phi}\right) \leq T(r,\Phi) + O(1)$$
$$\leq N(r,\Phi) + S(r,f) + S(r,g), \tag{2.2}$$

where $N_E^{1)}\left(r, \frac{1}{f^{(k)}-1}\right)$ denotes the counting function of common 1-points of $f^{(k)}$ and $g^{(k)}$.

According to our assumption, $\Phi(z)$ has simple poles only at zeros of $f^{(k+1)}$, $f^{(k)}-1$ and $g^{(k+1)}$, $g^{(k)}-1$ as well as poles of f and g.
From Lemma 2.1, we have

$$T(r,f) + T(r,g) \leq \overline{N}(r,f) + \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right)$$
$$+\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right)$$
$$-N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,f) + S(r,g). \tag{2.3}$$

Obviously,

$$N\left(r, \frac{1}{f^{(k)}-1}\right) \leq T(r, f^{(k)}) + 0(1) \leq T(r,f) + k\overline{N}(r,f) + S(r,f). \tag{2.4}$$

If $l \geq 2$, we have

$$N(r,\Phi) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}_{(l+1}\left(r, \frac{1}{f^{(k)}-1}\right)$$
$$+N_0\left(r, \frac{1}{f^{(k+1)}}\right) + N_0\left(r, \frac{1}{g^{(k+1)}}\right), \tag{2.5}$$

and

$$\overline{N}_{(l+1}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right)$$
$$\leq N_{1)}\left(r, \frac{1}{g^{(k)}-1}\right) + N\left(r, \frac{1}{f^{(k)}-1}\right). \tag{2.6}$$

From (2.2)−(2.6) we deduce that

$$T(r,g) \leq (k+2)\overline{N}(r,f) + 2\overline{N}(r,g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_{k+1}\left(r, \frac{1}{f}\right)$$
$$+ N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite linear measure such that $T(r,F) \leq T(r,G)$ for $r \in I$. Hence

$$T(r,g) \leq [(k+2)(1-\Theta(\infty,f)) + 2(1-\Theta(\infty,g)) + (1-\Theta(0,f))$$

$$+\,(1-\Theta(0,g))+(1-\delta_{k+1}(0,f))+(1-\delta_{k+1}(0,g))+\varepsilon]\,T(r,g)+S(r,g),$$

for $r \in I$ and $0<\varepsilon<\Delta_1-(k+7)$, that is $[\Delta_1-(k+7)-\varepsilon]\,T(r,g)\le S(r,g)$.
ie.,

$$\Delta_1 \le (k+7), \tag{2.7}$$

If $l=1$, then

$$N(r,\Phi)\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{g^{(k+1)}}\right). \tag{2.8}$$

Obviously,

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\le N_E^{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right). \tag{2.9}$$

Thus, we deduce from (2.2)–(2.4), (2.8) and (2.9) that

$$T(r,g)\le (k+2)\overline{N}(r,f)+2\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_{k+1}\left(r,\frac{1}{f}\right)$$
$$+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)+S(r,f)+S(r,g). \tag{2.10}$$

Note that $l=1$, from Lemma 2.3, we have

$$\overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)\le \overline{N}\left(r,\frac{1}{f^{(k+1)}}\right)=N_1\left(r,\frac{1}{f^{(k+1)}}\right)$$
$$\le N_{k+2}\left(r,\frac{1}{f}\right)+(k+1)\overline{N}(r,f)+S(r,f). \tag{2.11}$$

The inequality (2.10) together with (2.11) yields

$$T(r,g)\le (2k+3)\overline{N}(r,f)+2\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_{k+1}\left(r,\frac{1}{f}\right)$$
$$+N_{k+1}\left(r,\frac{1}{g}\right)+N_{k+2}\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g).$$

Hence

$$T(r,g)\le [(2k+3)(1-\Theta(\infty,f))+2(1-\Theta(\infty,g))+(1-\Theta(0,f))$$
$$+(1-\Theta(0,g))+(1-\delta_{k+1}(0,f))+(1-\delta_{k+1}(0,g))+(1-\delta_{k+2}(0,f))$$
$$+\varepsilon]\,T(r,g)+S(r,g),$$

for $r\in I$ and $0<\varepsilon<\Delta_2-(2k+9)$, that is $[\Delta_2-(2k+9)-\varepsilon]\,T(r,g)\le S(r,g)$,
ie.,

$$\Delta_2\le (2k+9). \tag{2.12}$$

If $l=0$, i.e., $f^{(k)}$ and $g^{(k)}$ share 1 IM, at this circumstance, we have

$$N(r,\Phi)\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+N_0\left(r,\frac{1}{g^{(k+1)}}\right). \tag{2.13}$$

From Lemma 2.4, we have

$$\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+2\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)\le \overline{N}(r,f)+2\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f^{(k)}}\right)$$
$$+2\overline{N}\left(r,\frac{1}{g^{(k)}}\right)+S(r,f)+S(r,g). \tag{2.14}$$

From Lemma 2.3, we can deduce that

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right)+2\overline{N}\left(r,\frac{1}{g^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)+2\ N_1\left(r,\frac{1}{g^{(k)}}\right)$$
$$\le N_{k+1}\left(r,\frac{1}{f}\right)+2N_{k+1}\left(r,\frac{1}{g}\right)+k\overline{N}(r,f)+2k\overline{N}(r,g)+S(r,f)+S(r,g). \tag{2.15}$$

When $l=0$, we can get

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)\le N_E^{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+N\left(r,\frac{1}{f^{(k)}-1}\right).$$

From (2.2)−(2.4) and (2.13)−(2.15) and the above inequality, we can obtain

$$T(r,g)\le (2k+3)\overline{N}(r,f)+(2k+4)\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)$$
$$+2N_{k+1}\left(r,\frac{1}{f}\right)+3N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \tag{2.16}$$

In the same way, we can also get

$$T(r,g)\le[(2k+3)(1-\Theta(\infty,f))+(2k+4)(1-\Theta(\infty,g))+(1-\Theta(0,f))$$
$$+(1-\Theta(0,g))+2(1-\delta_{k+1}(0,f))+3(1-\delta_{k+1}(0,g))+\varepsilon]T(r,g)+S(r,g),$$

for $r\in I$ and $0<\varepsilon<\Delta_3-(4k+13)$, that is $[\Delta_3-(4k+13)-\varepsilon]T(r,g)\le S(r,g)$, ie.,

$$\Delta_3\le(4k+13), \tag{2.17}$$

Hence, we get $\Phi(z)\equiv 0$, ie.,

$$\frac{f^{(k+2)}}{f^{(k+1)}}-2\frac{f^{(k+1)}}{f^{(k)}-1}=\frac{g^{(k+2)}}{g^{(k+1)}}-2\frac{g^{(k+1)}}{g^{(k)}-1}.$$

Integration yields

$$\frac{1}{f^{(k)}-1}\equiv\frac{bg^{(k)}+a-b}{g^{(k)}-1},$$

where a and b are two constants and $a\neq 0$. By using the same argument as in [13], we can obtain $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$, we here omit the detail. The proof of Lemma 2.6 is completed.

Lemma 2.7. *Let f and g be two non-constant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be a integers. Then*

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \neq 1.$$

Proof. Let

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1. \tag{2.18}$$

Let z_0 be a zero of f of order p_0. From (2.18) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (2.18), we obtain $np_0 - k = nq_0 + mq_0 + k$,

i.e., $n(p_0 - q_0) = mq_0 + 2k$.

which implies that $q_0 \geq \frac{n-2k}{m}$ and so we have $p_0 \geq \frac{n+m-2k}{m}$.

Let z_1 be a zero of $f-1$ of order p_1, then z_1 is a zero of $[f^n P(f)]^{(k)}$ of order $p_1 - k$. Therefore from (2.18) we obtain $p_1 - k = nq_1 + mq_1 + k$

i.e., $p_1 \geq (n+m)s + 2k$.

Let z_2 be a zero of f' of order p_2 that is not a zero of $fP(f)$, then from (2.18) z_2 is a pole of g of order q_2. Again by (2.18) we get $p_2 - (k-1) = nq_2 + mq_2 + k$

i.e., $p_2 \geq (n+m)s + 2k - 1$.

In the same manner as above, we have similar results for the zeros of $[g^n P(g)]^{(k)}$.

On other hand, suppose that z_3 is a pole of f. From (2.18), we get that z_3 is the zero of $[g^n P(g)]^{(k)}$.

Thus

$$\begin{aligned}
\overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right) \\
&\leq \frac{1}{p_0} N\left(r,\frac{1}{g}\right) + \frac{1}{p_1} N\left(r,\frac{1}{g-1}\right) + \frac{1}{p_2} N\left(r,\frac{1}{g'}\right) \\
&\leq \left[\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2}{(n+m)s+2k-1}\right] T(r,g) + S(r,g).
\end{aligned} \tag{2.19}$$

By second fundamental theorem and equation (2.19), we have

$$\begin{aligned}
T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}(r,f) \\
&\leq \frac{m}{n+m-2k} N\left(r,\frac{1}{f}\right) + \frac{1}{(n+m)s+2k} N\left(r,\frac{1}{f-1}\right) \\
&\quad + \left[\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2}{(n+m)s+2k-1}\right] T(r,g) + S(r,g) + S(r,f).
\end{aligned}$$

$$T(r,f) \leq \left[\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k}\right] T(r,f)$$

$$+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g)+S(r,g)+S(r,f). \quad (2.20)$$

Similarly, we have

$$T(r,g) \le \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\right]T(r,g)$$
$$+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,f)+S(r,g)+S(r,f). \quad (2.21)$$

Adding (2.20) and (2.21) we get

$$T(r,f)+T(r,g) \le \left[\frac{2m}{n+m-2k}+\frac{2}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]\{T(r,f)+T(r,g)\}$$
$$+S(r,g)+S(r,f).$$

which is a contradiction. Thus Lemma proved.

3. Proofs of the Theorems

In this section we present the proofs of the main results.

Proof of Theorem 1.1. Let $F=f^nP(f)$ and $G=g^nP(g)$.

Consider

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f^nP(f)}\right)\le\frac{1}{s(n+m)}N\left(r,\frac{1}{F}\right)\le\frac{2}{s(n+m)}[T(r,F)+O(1)].$$

$$\Theta(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)}\ge 1-\frac{2}{s(n+m)}. \quad (3.1)$$

Similarly,

$$\Theta(0,G)\ge 1-\frac{2}{s(n+m)}. \quad (3.2)$$

$$\Theta(\infty,F)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,F)}{T(r,F)}\ge 1-\frac{1}{s(n+m)}. \quad (3.3)$$

Similarly,

$$\Theta(\infty,G)\ge 1-\frac{1}{s(n+m)}. \quad (3.4)$$

Consider

$$N_{k+1}\left(r,\frac{1}{F}\right)=N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)=(k+1)\overline{N}\left(r,\frac{1}{f^nP(f)}\right)\le\frac{(k+1)}{s(n+m)}[T(r,F)+O(1)].$$

Next, we have

$$\delta_{k+1}(0,F)=1-\limsup_{r\to\infty}\frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)}\ge 1-\frac{(k+1)}{s(n+m)}. \quad (3.5)$$

Similarly,

$$\delta_{k+1}(0,G) \geq 1 - \frac{(k+1)}{s(n+m)}. \tag{3.6}$$

Case(i) If $l \geq 2$ and from (3.1) to (3.6) and also from Lemma 2.6, we get

$$\begin{aligned}\Delta_1 &= (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g)\\ &> (k+8) - \frac{3k+10}{s(n+m)}\end{aligned}$$

Since $s(n+m) > 3k+10$, we get $\Delta_1 > k+7$.

Therefore, by Lemma 2.6, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

If $F^{(k)}G^{(k)} \equiv 1$, that is

$$[f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0)]^{(k)} [g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0)]^{(k)} \equiv 1, \tag{3.7}$$

then by Lemma 2.7 we can get a contradiction.

Hence, we deduce that $F \equiv G$, that is

$$f^n(a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0) = g^n(a_m g^m + a_{m-1} g^{m-1} + \cdots + a_1 g + a_0). \tag{3.8}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.8) we obtain

$$a_m g^{n+m}(h^{n+m}-1) + a_{m-1} g^{n+m-1}(h^{n+m-1}-1) + \cdots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-1} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant , then we know (3.8) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Case(ii) If $l = 1$ and from (3.1) to (3.6) and also from Lemma 2.6, we get

$$\begin{aligned}\Delta_2 &= (2k+3)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) + \delta_{k+2}(0,f)\\ &> (2k+10) - \frac{5k+13}{s(n+m)}\end{aligned}$$

Since $s(n+m) > 5k+13$, we get $\Delta_2 > 2k+9$.

By continuing as in case(i), we get case(ii).

Case(iii) If $l = 0$ and from (3.1) to (3.6) and also from Lemma 2.6, we get

$$\begin{aligned}\Delta_3 &= (2k+3)\Theta(\infty,f) + (2k+4)\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + 2\delta_{k+1}(0,f) + 3\delta_{k+1}(0,g)\\ &> (4k+14) - \frac{9k+16}{s(n+m)}\end{aligned}$$

Since $s(n+m) > 9k+16$, we get $\Delta_2 > 4k+13$.

By continuing as in case(i), we get case(iii).

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since f and g are entire functions we have $\overline{N}(r,f) = \overline{N}(r,g) = 0$. Proceeding as in the proof of Theorem 1.1 we can easily prove Theorem 1.2.

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