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A NEW CLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION WITH VARYING ARGUMENT

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Abstract. In this paper, we introduce a new class of analytic functions defined by convolution with varying argument and provide convolution properties and integral means inequalities for functions belonging to this new class. Some of our results generalize previously known results.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$. Also, by $T_\gamma (\gamma \in R)$ we denote the class of functions $f(z) \in \mathscr{A}$ of the form (1.1) for which all of non-vanishing coefficients satisfy the condition

$$\arg(a_n) = \pi + (1-n)\gamma \quad (n = 2, 3, \ldots). \tag{1.2}$$

For $\gamma = 0$ we obtain the class T_0 of functions with negative coefficients. Moreover, we define

$$T = \bigcup_{\gamma \in R} T_\gamma.$$

The class T was introduced by Silverman [1] (see also [2], [3] and [18]). It is called the class of functions with varying argument of coefficients.

If f of the form (1.1) and $g(z) = z + b_2 z^2 + \cdots + b_j z^j + \cdots$ are two functions in $\mathscr{A}$, then the convolution of f and g is denoted by $f * g$ and is given by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j. \tag{1.3}$$

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For two functions f and g analytic in U, we say that the function f is subordinate to g in U, and write

$$f(z) \prec g(z) \ (z \in U).$$

If there exists a Schwarz function ω, which (by definition) is analytic in U, $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$ $(z \in U)$. Indeed it is known that

$$f(z) \prec g(z) \ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence [4, p.4]:

$$f(z) \prec g(z) \ (z \in U) \Leftrightarrow f(0) = g(0) \ and \ f(U) \subset g(U).$$

Definition 1. A sequence $\{b_j\}_{j=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{j=1}^{\infty} a_j b_j z^j \prec f(z) \ (z \in U; \ a_1 = 1). \tag{1.4}$$

Now we define the following new class of analytic functions, and obtain some interesting results. Let $U(\phi, \psi; \alpha, A, B)$ denote the subclass of $\mathscr{A}$ consisting of function $f(z)$ which satisfy the following inequality:

$$\frac{f(z) * \phi(z)}{f(z) * \psi(z)} - \alpha \left| \frac{f(z) * \phi(z)}{f(z) * \psi(z)} - 1 \right| \prec \frac{1 + Az}{1 + Bz} \ (\alpha \geq 0, -1 \leq B < A \leq 1), \tag{1.5}$$

where

$$\phi(z) = z + \sum_{j=2}^{\infty} \mu_j z^j \ and \ \psi(z) = z + \sum_{j=2}^{\infty} \eta_j z^j$$

are analytic in U such that $f(z) * \psi(z) \neq 0, \mu_j > \eta_j \geq 0 \ (j \geq 2)$.

For suitable choices of ϕ, ψ and by specializing the parameters α, A, B involved in the class $U(\phi, \psi; \alpha, A, B)$, we also obtain the following subclasses which were studied in many earlier works:

(1) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, A, B) = S^*(A, B)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, A, B) = K(A, B)$ (Janowski. W [5] and K. S. Padmanabhan et al. [6]). For example, we have $S^*(1, -1) = S^*$ (starlike functions); $K(1, -1) = K$ (convex functions);

(2) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1, -1) = US(\alpha)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1, -1) = UK(\alpha)$ (A. W. Goodman [7], W. C. Ma, D. Minda [8] and F. Ronning [9]);

(3) $U(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, 1-2\beta, -1) = US(\alpha, \beta)$ and $U(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1-2\beta, -1) = UK(\alpha, \beta)$ (S. S. Hams et al. [10] and S. Shams et al.[11]);

(4) $U(z+\sum_{j=2}^{\infty} j^{n+1}z^j, z+\sum_{j=2}^{\infty} j^n z^j; \alpha, 1-2\beta, -1) = US_n(\alpha, \beta)$
(S.Shams et al. [13], R. Bharati et al. [14] and Wei-Ping Kuang, et al. [15]);

(5) $U(z+\sum_{j=2}^{\infty} j^m z^j, z+\sum_{j=2}^{\infty} j^n z^j; \alpha, 1-2\beta, -1) = US_{m,n}(\alpha, \beta)$
$(0 \le \alpha,\ 0 \le \beta < 1)$ (S. S. Eker and S. Owa [16]);

(6) $U(z+\sum_{j=2}^{\infty} j^m z^j, z+\sum_{j=2}^{\infty} j^n z^j; \alpha, 1-2\beta, -1) = US_{m,n}(\alpha, \beta)$
$(0 \le \alpha,\ 0 \le \beta < 1)$ (H.M. Srivastava and S.S. Eker [17]);

(7) $U(\phi, \psi; 0, A, B) = U(\phi, \psi; A, B)$ (J.Dziok [18]) ;

(8) (6) $U(z+\sum_{j=2}^{\infty} j^m z^j, z+\sum_{j=2}^{\infty} j^n z^j; \alpha, A, B) = US_{m,n}(\alpha, A, B)$
$(0 \le \alpha,\ 0 \le \beta < 1)$ (Shu-Hai Li and Huo Tang [19]).

Now, we define the classes of functions with varying argument of coefficients related to the class $U(\phi, \psi; \alpha, A, B)$. Let us denote

$$TU_\gamma(\phi, \psi; \alpha, A, B) = T_\gamma \cap U(\phi, \psi; \alpha, A, B),$$
$$TU(\phi, \psi; \alpha, A, B) = T \cap U(\phi, \psi; \alpha, A, B).$$

In this paper, we establish a theorem concerning the subordination results of functions in the class $TU_\gamma(\phi, \psi; \alpha, A, B)$.

2. Main results

To prove our main results, we need the following lemmas.

Lemma 1 ([20])**.** *The sequence* $\{b_j\}_{j=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{j=1}^{\infty} b_j z^j\right\} > 0,\ (z \in U).$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $TU_\gamma(\phi, \psi; \alpha, A, B)$.

Lemma 2. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\sum_{j=2}^{\infty} \phi_j(\mu_j, \eta_j, \alpha, A, B)|a_j| \le A - B, \tag{2.1}$$

where

$$\phi_j(\mu_j, \eta_j, \alpha, A, B) = (1 + (1+|B|)\alpha)(\mu_j - \eta_j) + |B\mu_j - A\eta_j|. \tag{2.2}$$

For some $\alpha \geq 0, -1 \leq B < A \leq 1, \mu_j > \eta_j \geq 0$ $(j \geq 2)$, then $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$.

Proof. Suppose that (2.1) is true for $\alpha \geq 0, -1 \leq B < A \leq 1$. For $f(z) \in \mathscr{A}$, let us define the function $p(z)$ by

$$p(z) = \frac{f(z)*\phi(z)}{f(z)*\psi(z)} - \alpha\left|\frac{f(z)*\phi(z)}{f(z)*\psi(z)} - 1\right|.$$

It suffices to show that

$$\left|\frac{p(z)-1}{A-Bp(z)}\right| < 1 \ (z \in U). \tag{2.3}$$

We note that

$$\begin{aligned}
&\left|\frac{p(z)-1}{A-Bp(z)}\right| \\
&= \left|\frac{f(z)*\phi(z) - \alpha e^{i\theta}|f(z)*\phi(z) - f(z)*\psi(z)| - f(z)*\psi(z)}{Af(z)*\psi(z) - B(f(z)*\phi(z) - \alpha e^{i\theta}|f(z)*\phi(z) - f(z)*\psi(z)|)}\right| \\
&= \left|\frac{(f(z)*\phi(z) - f(z)*\psi(z)) - \alpha e^{i\theta}|f(z)*\phi(z) - f(z)*\psi(z)|}{(A-B)f(z)*\psi(z) - B((f(z)*\phi(z) - f(z)*\psi(z)) - \alpha e^{i\theta}|f(z)*\phi(z) - f(z)*\psi(z)|)}\right| \\
&= \left|\frac{\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1} - \alpha e^{i\theta}|\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1}|}{(A-B) - \sum_{j=2}^{\infty}(B\mu_j - A\eta_j)a_j z^{j-1} - \alpha B e^{i\theta}|\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1}|}\right| \\
&\leq \frac{\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j||z|^{j-1} + \alpha|e^{i\theta}|\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j||z|^{j-1}}{(A-B) - \sum_{j=2}^{\infty}|B\mu_j - A\eta_j||a_j||z|^{j-1} - \alpha|B||e^{i\theta}|\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j||z|^{j-1}} \\
&\leq \frac{\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j| + \alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|}{(A-B) - \sum_{j=2}^{\infty}|B\mu_j - A\eta_j||a_j| - \alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|}
\end{aligned}$$

The last expression is bounded above by 1, if

$$\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j| + \alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j| \leq (A-B) - \sum_{j=2}^{\infty}|B\mu_j - A\eta_j||a_j| - \alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|$$

which is equivalent to our condition (2.1). This completes the proof of our theorem. □

Remark. Taking $\phi(z) = z + \sum_{j=2}^{\infty} j^m z^j$, $\psi(z) = z + \sum_{j=2}^{\infty} j^n z^j$ $(0 \leq \alpha,\ 0 \leq \beta < 1)$, we obtain the improved result of Theorem 1 in the paper [19].

Lemma 3. *Let $f(z)$ be a function of the form* (1.1), *with* (1.2). *Then $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$ if and only if the inequality* (2.1) *holds true.*

Proof. In view of Lemma 2, we need only show that each function $f(z)$ from the class $TU_\gamma(\phi,\psi;\alpha,A,B)$ satisfies the coefficient inequality (2.1). Let $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$, by (2.3) and (1.1), we obtain

$$\left|\frac{\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1} - \alpha e^{i\theta}|\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1}|}{(A-B) - \sum_{j=2}^{\infty}(B\mu_j - A\eta_j)a_j z^{j-1} - \alpha B e^{i\theta}|\sum_{j=2}^{\infty}(\mu_j-\eta_j)a_j z^{j-1}|}\right| < 1 \ \ (z \in U).$$

Therefore, putting $z = re^{i\gamma}$ $(0 \le \gamma < 1)$, and applying (1.2) we have

$$\frac{\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|r^{j-1}+\alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|r^{j-1}}{(A-B)-\sum_{j=2}^{\infty}|B\mu_j-A\eta_j||a_j|r^{j-1}-\alpha\sum_{j=2}^{\infty}|\mu_j-\eta_j||a_j|r^{j-1}}<1.$$

It is clear that the denominator of the left hand said cannot vanish for $0 \le r < 1$, thus, we obtain

$$\sum_{j=2}^{\infty}((1+(1+|B|)\alpha)(\mu_j-\eta_j)+|B\mu_j-A\eta_j|)|a_j|r^{j-1},$$

which, upon letting $r \to 1^-$, readily yields the assertion (2.1).

Since the condition (2.1) is independent of γ, Lemma 2 yields the following theorem.

From Lemma 3, we have the following Theorem 1 and Theorem 2.

Theorem 1. *If a function of $f(z)$ the form (1.1), $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$, then*

$$|a_j| \le \frac{A-B}{\phi_j(\mu_j,\eta_j,\alpha,A,B)} \quad (j=2,3,\ldots),$$

where $\phi_j(\mu_j,\eta_j,\alpha,A,B)$ is defined by (2.2), the result is sharp. The functions $f_{j,\gamma}(z)$ of the form

$$f_{j,\gamma}(z) = z - \frac{A-B}{\phi_j(\mu_j,\eta_j,\alpha,A,B)}e^{i(1-j)\gamma}z^j \quad (z \in U;\, j=2,3,\ldots) \tag{2.4}$$

are the extreme functions.

Theorem 2. *Let a function $f(z)$ of the form (1.1) belong to class $TU_\gamma(\phi,\psi;\alpha,A,B)$. If the sequence $\{\phi_j(\mu_j,\eta_j,\alpha,A,B)\}_{j=1}^{\infty}$ defined by (2.2) satisfies the inequality*

$$\phi_2(\mu_2,\eta_2,\alpha,A,B) \le \phi_j(\mu_j,\eta_j,\alpha,A,B) \quad (j=2,3,\ldots), \tag{2.5}$$

then

$$\sum_{j=2}^{\infty} a_j \le \frac{2(A-B)}{\phi_2(\mu_2,\eta_2,\alpha,A,B)}.$$

Moreover, if

$$j\phi_2(\mu_2,\eta_2,\alpha,A,B) \le 2\phi_j(\mu_j,\eta_j,\alpha,A,B) \quad (j=2,3,\ldots), \tag{2.6}$$

then

$$\sum_{j=2}^{\infty} ja_j \le \frac{2(A-B)}{\phi_2(\mu_2,\eta_2,\alpha,A,B)}.$$

Employing the technique used earlier by Attiya [21], Srivastava and Attiya [22] and M. K. Aouf [23], we prove:

Theorem 3. *Let $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$. Then*

$$\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}(f*g)(z) \prec g(z) \tag{2.7}$$

for every function g in K, and

$$\operatorname{Re} f(z) > -\frac{1}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}, \quad (z\in U). \tag{2.8}$$

The constant factor $\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}$ cannot be replaced by a larger number.

Proof. Let $f(z) \in TU_\gamma(\phi,\psi;\alpha,A,B)$ and suppose that $g(z) = z+\sum_{j=2}^{\infty} c_j z^j \in K$. Then

$$\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}(f*g)(z) = \frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}(z+\sum_{j=2}^{\infty} a_j c_j z^j)$$

Thus, by Definition 1 the subordination result (2.7) will hold true if the sequence

$$\left\{\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}a_j\right\}_{j=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+2\sum_{j=1}^{\infty}\frac{a_j z^j}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}\right\}, \quad (z\in U). \tag{2.9}$$

Thus, by (2.5) for $|z| = r < 1$, we have

$$\begin{aligned}
&\operatorname{Re}\left\{1+2\sum_{j=1}^{\infty}\frac{a_j z^j}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}\right\}\\
&= \operatorname{Re}\left\{1+\frac{z}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}+\sum_{j=2}^{\infty}\frac{a_j z^j}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}\right\}\\
&\geq 1-\frac{r}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}-\frac{1}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}\sum_{j=2}^{\infty}\phi_j(\mu_j,\eta_j,\alpha,A,B)|a_j|r^j\\
&> 1-\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}r-\frac{A-B}{(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}r>0
\end{aligned}$$

where we have also made use of assertion (2.1) of Lemma 2. Thus (2.9) holds true in U. This proves the inequality (2.7). The inequality (2.8) follows from (2.7) by taking the convex function

$$g(z) = \frac{z}{1-z} = z+\sum_{j=2}^{\infty} z^j.$$

Next we observe that the function $f_{2,\gamma}(z)$ of the form (2.4) belongs to the class $TU_\gamma(\phi,\psi;\alpha,A,B)$.

It is easily verified that

$$\min\left\{\mathrm{Re}\left(\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}f_{2,\gamma}(z)\right)\right\}=-\frac{1}{2}\quad (z\in U).$$

This shows that the constant $\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}$ cannot be replaced by any larger one.

Corollary 1. *Let $f(z)\in TU(\phi,\psi;\alpha,A,B)$. Then*

$$\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}(f*g)(z)\prec g(z) \tag{2.10}$$

for every function g in K, and

$$\mathrm{Re} f(z)>-\frac{1}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))},\quad (z\in U). \tag{2.11}$$

The constant factor $\frac{\phi_2(\mu_2,\eta_2,\alpha,A,B)}{2(A-B+\phi_2(\mu_2,\eta_2,\alpha,A,B))}$ cannot be replaced by a larger number.

Now, we discuss the integral means inequalities of functions $f(z)$ in $TU_\gamma(\phi,\psi;\alpha,A,B)$.

Lemma 4 (Littlewood [24])**.** *If $f(z)$ and $g(z)$ are analytic in U with $f(z)\prec g(z)$, then for $\mu>0$ and $z=re^{i\theta}$ $(0<r<1)$*

$$\int_0^{2\pi}|f(z)|^\mu d\theta\le\int_0^{2\pi}|g(z)|^\mu d\theta.$$

Theorem 4. *Let $f(z)$ in $TU_\gamma(\phi,\psi;\alpha,A,B)$ and suppose that $f_n(z)$ is defined by (2.4). If there exists an analytic function $\omega(z)$ given by*

$$\omega(z)=\frac{\phi_n(\mu_n,\eta_n,\alpha,A,B)}{(A-B)e^{i(1-n)\gamma}}\sum_{j=2}^{\infty}a_jz^{j-1}\quad (n=2,3,\dots) \tag{2.12}$$

then for $z=re^{i\theta}$ $(0<r<1)$ and $\mu>0$,

$$\int_0^{2\pi}|f(re^{i\theta})|^\mu d\theta\le\int_0^{2\pi}|f_n(re^{i\theta})|^\mu d\theta. \tag{2.13}$$

Proof. For $z=re^{i\theta}$ $(0<r<1)$, we must show that

$$\int_0^{2\pi}\left|1+\sum_{j=2}^{\infty}a_jz^{j-1}\right|^\mu d\theta\le\int_0^{2\pi}\left|1+\frac{(A-B)e^{i(1-n)\gamma}}{\phi_n(\mu_n,\eta_n,\alpha,A,B)}z^{n-1}\right|^\mu d\theta\quad (\mu>0).$$

Thus by applying Littlewood's subordination theorem, it would suffice to show that

$$1+\sum_{j=2}^{\infty}a_jz^{j-1}\prec 1+\frac{(A-B)e^{i(1-n)\gamma}}{\phi_n(\mu_n,\eta_n,\alpha,A,B)}z^{n-1}. \tag{2.14}$$

By setting

$$1+\sum_{j=2}^{\infty} a_j z^{j-1} = 1+\frac{(A-B)e^{i(1-n)\gamma}}{\phi_n(\mu_n,\eta_n,\alpha,A,B)}\omega(z),$$

we find that

$$\omega(z) = \frac{\phi_n(\mu_n,\eta_n,\alpha,A,B)}{(A-B)e^{i(1-n)\gamma}}\sum_{j=2}^{\infty} a_j z^{j-1},$$

which readily yields $\omega(0) = 0$.

Therefore, we have

$$|\omega(z)| = \left|\frac{\phi_n(\mu_n,\eta_n,\alpha,A,B)}{(A-B)e^{i(1-n)\gamma}}\sum_{j=2}^{\infty} a_j z^{j-1}\right| \le \frac{\phi_n(\mu_n,\eta_n,\alpha,A,B)}{A-B}\sum_{j=2}^{\infty} |a_j||z|^{j-1} \le |z| < 1,$$

by means of hypothesis and Lemma 4.

We remark in conclusion that, by suitably specializing the parameters involved in the results presented in this paper, we can deduce numerous further corollaries and consequences of each of these results.

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