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ON h-PURIFIABLE SUBMODULE OF QTAG-MODULE

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Abstract. Different concepts and decomposition theorems have been done for QTAG-modules by a number of authors. The concept of quasi h-pure submodules were introduced and different characterizations were obtained in [5]. The purpose of this paper is to obtain the relation between purifiability of a submodule and quasi h-pure submodules. Further we obtained results which shows that purifiability of a submodule is very much dependent on the purifiability of a h-pure and h-dense submodule of the given submodule.

0. Introduction

S. Singh [9] introduced the concept of QTAG-module and did different decomposition theorems. A module M_R is called QTAG-module if it satisfies the condition : Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Since the different concepts for QTAG-modules have been introduced by different authors and various results based on those concepts have been obtained. In [5], Mohd. Z. Khan and A. Zubair introduced the concept of quasi h-pure submodules and obtained various characterizations and their consequences. In this paper we continue the similar study in term of purifiability of submodules and obtained a characterization. We have also established a necessary and sufficient condition for a submodule to be h-purifiable.

1. Preliminaries

Rings considered here are with unity ($1 \neq 0$) and modules are unital QTAG-module. A module in which the lattice of its submodules is totally ordered is called a serial module; in addition if it has finite composition length it is called uniserial module. An element $x \in M$ is called uniform if xR is a non zero uniform (hence uniserial) submodule of M. If $x \in M$ is uniform then $e(x) = d(xR)$ (The composition length of xR), $H_M(x) = \sup\{d(yR/xR)/x \in yR$

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and $y \in M$ is uniform $\}$ are called exponent of x and height of x in M respectively. For any $n \geq 0, H_n(M) = \{x \in M / H_M(x) \geq n\}$. A submodule N of M is called h-pure in M if $H_k(N) = N \cap H_k(M)$ for all $k \geq 0$, N is h-neat in M if $H_1(N) = N \cap H_1(M)$. The module M is called h-divisible if $H_1(M) = M$. For other basic concepts of QTAG-module one may see [2, 3, 4, 7, 8, 9].

2. Purifiability

Firstly we recall the following:

Lemma A([2])**.** If A and B are any two uniserial submodules of a QTAG-module M such that $A \cap B \neq 0$ and $d(A) \leq d(B)$. Then there exists a monomorphism $\sigma : A \longrightarrow B$, which is identity on $A \cap B$.

Definition 2.1 ([8])**.** A submodule N of a QTAG-module M is called h-dense if M/N is h-divisible.

Definition 2.2 ([7])**.** A submodule N of a QTAG-module M is said to be almost h-dense in M if for every h-pure submodule K of M containing N, M/K is h-divisible.

Now we restate the following :

Theorem 2.3 ([7], Theorem 5)**.** *A submodule N of a QTAG-module M is almost h-dense in M if and only if $N + H_n(M) \supseteq Soc(H_{n-1}(M))$ for all $n \geq 1$.*

Now before defining the h-purifiable submodule, we would like to adopt the following notations and results from [5].

Notation 2.4 ([5])**.** For any non-negative integer t and for a submodule N of a QTAG-module M, we denote by $N^t(M)$ the submodule $\big(N + H_{t+1}(M)\big) \cap Soc\big(H_t(M)\big)$, by $N_t(M)$ the submodule $\big(N \cap Soc(H_t(M))\big) + Soc\big(H_{t+1}(M)\big)$ and by $Q_t(M,N) = N^t(M)\big/N_t(M)$.

It is trivial to see that

$$\begin{aligned} N^t(M) &= \big(N + H_{t+1}(M)\big) \cap Soc\big(H_t(M)\big) \\ &= Soc(N \cap H_t(M) + H_{t+1}(M)) \end{aligned}$$

and

$$\begin{aligned} N_t(M) &= \big(N \cap Soc(H_t(M))\big) + Soc\big(H_{t+1}(M)\big) \\ &= (Soc(N))^t(M) \end{aligned}$$

Theorem 2.5 ([5], Theorem 4.2)**.** *If N and K are submodules of QTAG-module M such that $N \subseteq K$ and K is h-pure in M, then the module $Q_n(M,N)$ and $Q_n(K,N)$ are isomorphic, for all n.*

Theorem 2.6 ([5], Theorem 4.3)**.** *If N is h-neat submodule of M, then N is h-pure in M if and only if $Q_n(M,N) = 0$ for all $n \in Z^+$.*

Now we define h-purifiable submodule:

Definition 2.7. A submodule N of a QTAG-module M is called h-purifiable in M if there exists a submodule K of M minimal among the h-pure submodules of M containing N.

Such K is called h-pure hull of N in M.

Now we restate the following:

Theorem 2.8 ([6])**.** *A submodule N of a QTAG-module M is h-purifiable in M if and only if there exists a h-pure submodule K of M such that $Soc(H_n(K)) \subseteq N \subseteq K$ for some $n \in Z^+$.*

Proposition 2.9. *If N is a h-purifiable submodule of a QTAG-module M; then there exists $m \in Z^+$ such that $Q_n(M,N) = 0$ for all $n \geq m$.*

Proof. If N itself is an h-pure submodule, then by [5, Theorem 4.7], $Q_n(M,N) = 0$ for all $n \geq 0$. Now appealing Theorem 2.8, we get an h-pure submodule K of M and $m \in Z^+$ such that $Soc(H_m(K)) \subseteq N \subseteq K$. Now for $n \geq m$ it is trivial to see that $N^n(K) = \big(N + H_{n+1}(K)\big) \cap Soc\big(H_n(K)\big) = Soc(H_n(K)) = N_n(K)$. Hence, $Q_n(K,N) = 0$ for all $n \geq m$. Therefore from Theorem 2.5, we get $Q_n(M,N) = 0$ for all $n \geq m$.

Now we generalize [1, Theorem 66.3] and is of very interesting nature.

Theorem 2.10. *If M is a QTAG-module then every h-dense subsocle of M supports an h-pure and h-dense submodule.*

Proof. Let S be a subsocle of M and S be h-dense; then $Soc(M) = S + Soc(H_k(M))$ for all $k \in Z^+$. Let N be maximal with the property $Soc(N) = S$. Firstly we show that N is h-neat submodule of M. Let x be a uniform element in $N \cap H_1(M)$, then for a uniform element $y \in M$, we have $d(yR/xR) = 1$. If $y \in N$, then $x \in H_1(N)$. Let $y \notin N$ then $S \underset{\neq}{\subset} Soc(N + yR)$. Hence, there exists a uniform element $z \in Soc(N + xR)$ such that $z \notin S$ and $z = u + yr$ where $u \in N$ and $r \in R$. Trivially $yrR = yR$, hence without any loss of generality we can assume $z = u + y$. Define $\eta : yR \longrightarrow uR$ such that $\eta(yr) = ur$. Let $yr = 0$, then $zr = ur$. If $zrR = zR$ then $z \in S$, a contradiction, therefore $zr = 0$ and we get $ur = 0$, consequently η is a well defined epimorphism. Therefore, uR is a uniform submodule. Since $u+y \in Soc(M)$, $H_1(uR) =$

$H_1(yR)$, but xR is a maximal submodule of M; hence $H_1(yR) = xR$ and we get $x \in H_1(N)$. Thus, $N \cap H_1(M) = H_1(N)$. Now suppose $N \cap H_n(M) = H_n(N)$ and let x be a uniform element in $N \cap H_{n+1}(M)$; then $d(yR/xR) = 1$ for some uniform element $y \in H_n(M)$. Since N is h-neat in M, there is a uniform element $y' \in N$ such that $d(y'R/xR) = 1$. Hence by Lemma A, there is an isomorphism $\sigma : yR \longrightarrow y'R$ which is identity on xR. The map $\eta : yR \longrightarrow (y - y')R$ where $\sigma(y) = y'$ is an epimorphism with $xR \subseteq \operatorname{Ker} \eta$. Hence, $e(y - y') \leq 1$ and we get $y - y' \in Soc(M) = S + Soc(H_n(M))$. Therefore, $y - y' = s + t$ for some $s \in S, t \in H_n(M)$. Consequently, $y - t = y' + s \in N \cap H_n(M) = H_n(N)$. Since $y - y' - s \in Soc(M)$, $H_1(yR) = H_1((y' + s)R) \subseteq H_{n+1}(N)$. Hence, $x \in H_{n+1}(N)$. Therefore, N is h-pure submodule of M.

Now let $\bar{x} \in Soc(M/N) = (Soc(M) + N)/N$ be a uniform element; then by [Lemma 3.9, 9] there exists a uniform element $x' \in M$ such that $\bar{x} = \bar{x'}$ and $e(x') = 1$. Since $Soc(M) = S + Soc(H_k(M))$ for all k, we get $\bar{x} \in H_k(M/N)$ for every k. Hence, $\bar{x} \in \cap_{k=1}^{\infty} H_k(M/N)$ and appealing to [9, Theorem 3.11], we get M/N is h-divisible. Hence, N is h-dense in M.

Observation: Using the notations used earlier, the h-purity can be established as: Since $Soc(M) = Soc(N) + Soc(H_k(M))$ for all $k \in Z^+$, it is easy to see that $N^n(M) = N_n(M)$ and $Q_n(M, N) = 0$ for all $n \in Z^+$. Since N is h-neat, therefore by [5, Theorem 4.7], N is h-pure in M.

Theorem 2.11. *If N is almost h-dense submodule of a QTAG-module M. Then N is h-purifiable in M if and only if there exists $m \in Z^+$ such that $Q_n(M, N) = 0$ for all $n \geq m$.*

Proof. Let N be h-purifiable then by Theorem 2.9, we get $Q_n(M, N) = 0$ for all $n \geq m$. Conversely, suppose that $Q_n(M, N) = 0$ for all $n \geq m$ and N is almost h-dense in M. Then $N^n(M) = N_n(M) = Soc(N \cap H_n(M)) + Soc(H_{n+1}(M))$. Since N is almost h-dense in M, therefore by Theorem 2.3, we get $Soc(H_n(M)) = Soc(N \cap H_n(M)) + Soc(H_{n+1}(M))$ for all $n \geq m$. Therefore, $Soc(N \cap H_m(M))$ is h-dense subsocle of $H_m(M)$. Now appealing to Theorem 2.10, we can find an h-pure submodule K of $H_m(M)$ such that $Soc(K) \subseteq N \cap H_m(M) \subseteq K$. It is easy to see that $H_m(M)/K$ is h-divisible submodule of M/K and $H_m(M)/K \cap (N + K)/K = 0$. Hence there exists a submodule T/K such that $(N + K)/K \subseteq T/K$ and $M/K = H_m(M)/K \oplus T/K$. Now by [Proposition 2.5, 4], T is h-pure submodule of M. Trivially $T \cap H_m(M) = K$, but $T \cap H_m(M) = H_m(T)$; so $H_m(T) = K$. Hence, $Soc(H_m(K)) \subseteq Soc(K) \subseteq N$. Hence by Theorem 2.8, we get N to be h-purifiable.

3. Role of h-pure and h-dense submodules

In this section we show that h-purifiability of a submodule depends upon the h-purifiability of an h-pure and h-dense submodule of the given submodule.

Firstly we prove the following results for obtaining a necessary and sufficient condition for h-purifiability.

Theorem 3.1. *If B is an h-pure and h-dense submodule of a submodule K of a QTAG-module M, then $Q_n(M,N) = Q_n(M,B)$ for all $n \in Z^+$.*

Proof. Since B is h-dense in K, then we have $K = B + H_{n+1}(K)$ for all $n \geq 0$ and hence, $K + H_{n+1}(M) = B + H_{n+1}(M)$. Therefore, $K^n(M) = B^n(M)$ for all $n \geq 0$.
Further, $K_n(M) = (Soc(K))^n(M) = (Soc(K) + H_{n+1}(M)) \cap Soc(H_n(M)$. Now appealing to [3, Prop.6], we get

$$\begin{aligned} K_n(M) &= \Big(Soc(B) + Soc(H_{n+1}(K)) + H_{n+1}(M)\Big) \cap Soc(H_n(M)) \\ &= \Big(Soc(B) + H_{n+1}(M)\Big) \cap Soc(H_n(M)) \\ &= B_n(M) \end{aligned}$$

Hence, $Q_n(M,K) = Q_n(M,B)$.

Proposition 3.2. *If B is a h-pure and h-dense submodule of a submodule K of a QTAG-module M. If K is h-purifiable in M, then B is h-purifiable in M.*

Proof. Let T be a h-pure hull of K in M. Since B is h-dense in K we get, K/B is h-divisible, so $T/B = K/B \oplus L/B$. Appealing to [Proposition 2.5, 4] we get, L to be h-pure submodule of T and hence L is h-pure in M. Let N be a h-pure submodule of M such that $B \subseteq N \subseteq L$. Then we claim that $K+N$ is a h-pure submodule of M. Since $K = B+H_n(K)$, we have $K+N = H_n(K)+N$. Therefore,

$$\begin{aligned} (K+N) \cap H_n(M) &= (H_n(K)+N) \cap H_n(M) \\ &= H_n(K) + (N \cap H_n(M)) \\ &= H_n(K) + H_n(N) \\ &= H_n(K+N) \end{aligned}$$

for all $n \geq 0$.

Since T is a h-pure hull of K in M, we have $K + N = T$ and

$$L = (K+N) \cap L = N + (K \cap L) = N + B = N$$

Therefore, L is a h-pure hull of B in M.

Proposition 3.3. *If B is a h-pure and h-dense submodule of a submodule K of a QTAG-module M and if N be a h-pure hull of B in M and $Soc(N) = Soc(B)$, then $K + N$ is a h-pure hull of K in M.*

Proof. Since K/B is h-divisible, we have $K = B + H_n(K)$. Now $K + N = B + H_n(K) + N = N + H_n(K)$ and hence $(K + N) \cap H_n(M) = (N + H_n(K)) \cap H_n(M) = H_n(K) + N \cap H_n(M) = H_n(K) + H_n(N) = H_n(K + N)$ for all $n \geq 0$. Therefore, $K + N$ is h-pure submodule of M. Since $Soc(N) = Soc(B)$, so $Soc(K \cap N) = Soc(B)$, so $N \cap K$ is an essential extension of B in K. Since h-pure submodules have no proper essential extensions, therefore we get, $K \cap N = B$. Now we show that $Soc(K + N) = Soc(K)$, which will yield that $N + K$ is h-pure hull of K in M. Using [Lemma 1, 2] we can proceed as: If $x \in Soc(K + N)$ then $H_1(xR) = 0$ and $x = k + t$ where $k \in K, t \in N$, then $H_1(tR) = H_1(kR) \subseteq N \cap K = B \cap H_1(K) = H_1(B)$. Hence, $H_1(tR) = H_1(kR) = H_1(bR)$ for $b \in B$. Hence, $k - b \in Soc(K)$ and $t + b, t - b \in Soc(N) = Soc(B)$. Hence $x = k - b + b + t \in Soc(K)$ and we get $Soc(K + N) = Soc(K)$.

Proposition 3.4. *If K is a h-pure hull of a submodule N of a QTAG-module M such that $Soc(K) \neq Soc(N)$. Then there exists $m \in Z^+$ such that $Q_m(M, N) \neq 0$.*

Proof. From Theorem 2.8 and Theorem 2.3, there exists $n \in Z^+$ such that $Soc(H_n(K)) \subseteq N$ and $Soc(H_t(K)) \subseteq N + H_{t+1}(K)$ for all $t \geq 0$. Since $Soc(K) \neq Soc(N)$, the smallest n such that $Soc(H_n(K)) \subseteq N$, we have $n \neq 0$.

Now taking $n = m - 1, N^m(K) = Soc(H_m(K))$ while $N_m(K) \subset N$. Therefore, $N^m(K) \neq N_m(K)$ but by [5, Theorem 2.1], $Q_m(M, N) \cong Q_m(M, K) \neq 0$. Hence, $Q_m(M, N) \neq 0$.

Proposition 3.5. *Let N be a submodule of a QTAG-module M. if N is h-purifiable in M, then $N \cap H_n(M)$ is h-purifiable in $H_n(M)$ for all $n \geq 0$. Conversely, if $N \cap H_n(M)$ is h-purifiable in $H_n(M)$ for some $n \geq 1$, then N is h-purifiable in M.*

Proof. Let K be h-pure hull of N in M, then trivially $H_n(K)$ is h-pure submodule $H_n(M)$ for all $n \in Z^+$. Also $H_n(K) = K \cap H_n(M) \supseteq N \cap H_n(M)$.

Now we claim that $H_n(K)$ is h-pure hull of $N \cap H_n(M)$ in $H_n(M)$. Let T be h-pure submodule of $H_n(M)$ such that $H_n(K) \supseteq T \supseteq N \cap H_n(M)$. Trivially $N \cap H_n(K) \subseteq N \cap H_n(M)$ and $N \cap H_n(K) \supseteq T \cap N \supseteq N \cap H_n(M)$; consequently $H_n(K) \supseteq T \supseteq N \cap H_n(M) = N \cap H_n(K)$. Now appealing to [Theorem 4.12, 5], we can extend $N + T$ to an h-pure submodule D of K such that $D \cap H_n(K) = T$ (we can note that $(N + T) \cap H_n(K) = T + N \cap H_n(K) = T$). Thus, $D = K$ and we get $H_n(K) = T$. Hence, $H_n(K)$ is h-pure hull of $N \cap H_n(M)$ in $H_n(M)$. Conversely, let $N \cap H_n(M)$ be h-purifiable in $H_n(M)$ and T be h-pure hull of $N \cap H_n(M)$ in $H_n(M)$. Then as done above $(N + T) \cap H_n(M) = T$ and $N + T$ can be extended to an h-pure submodule K of M such that $K \cap H_n(M) = T$. Clearly $T = H_n(K)$. Appealing to Theorem 2.8 there exists $m \in Z^+$ such that $Soc(H_m(T)) \subseteq H_n(M)$; so $Soc(H_{m+n}(K)) \subseteq N \subseteq K$. Hence by Theorem 2.8, N is h-purifiable in M.

Theorem 3.6. *If N is a submodule of a QTAG-module M. Then N is h-purifiable if and only if all basic submodules of N are h-purifiable.*

Proof. Let all basic submodules of N be h-purifiable. Then by Theorem 3.1 and Theorem 2.11, there exists $m \in Z^+$ such that $Q_n(M,N) = 0$ for all $n \geq m$. Hence $Q_n(H_m(M), N \cap H_m(M)) = 0$ for all $n \geq 0$. Let B be a basic submodule of $N \cap H_n(M)$; then $N/B = (N \cap H_n(M))/B \oplus T/B$ and we get T to be h-pure in N (see [Proposition 2.5, 4]) also $T/B \cong N/(N \cap H_m(M)) \cong (N + H_m(M))/H_m(M)$ is trivially bounded. Hence, T is also a direct sum of uniserial modules and we get T to be a basic submodule of N. As given, T is h-purifiable in M, therefore $T \cap H_m(M) = B$ is h-purifiable in $H_m(M)$ by Proposition 3.5; consequently B is h-purifiable basic submodule of $N \cap H_m(M)$ in $H_m(M)$, and $Q_n(H_m(M), B) = 0$ for all $n \geq 0$. Now let L be a h-pure hull of B in $H_m(M)$, then $Q_n(L,B) = 0$ for all $n \geq 0$, and by Proposition 3.4, $Soc(L) = Soc(B)$. Hence by Proposition 3.3, $N \cap H_m(M)$ is h-purifiable in $H_m(M)$ and so by Proposition 3.5, N is h-purifiable in M. The converse follows from Theorem 3.2.

Lastly we prove the following result which is of particular interest.

Theorem 3.7. *If N is almost h-dense submodule of a QTAG-module M and K is h-pure hull of $Soc(N)$. Then $Q_n(M,N) \cong (Soc(H_n(M)) + K) \Big/ (Soc(H_{n+1}(M)) + K)$ for all $n \in Z^+$.*

Proof. As N is almost h-dense in M, then appealing to Theorem 2.3, we have $N^n(M) = Soc(H_n(M))$ Since K is h-pure hull of $Soc(N)$ in $M, Soc(K) = Soc(N)$. Therefore, $N_n(M) = Soc(N \cap H_n(M)) + Soc(H_{n+1}(M)) = Soc(H_n(K)) + Soc(H_{n+1}(M))$. So we get $Q_n(M,N) = Soc(H_n(M))/(Soc(H_n(K)) + Soc(H_{n+1}(M)))$. Now we define a map $\eta : Q_n(M,N) \longrightarrow (Soc(H_n(M)) + K)/(Soc(H_{n+1}(M)) + K)$ given as $\eta(x + Soc(H_n(K)) + Soc(H_{n+1}(M))) = x + Soc(H_{n+1}(M)) + K$. Then trivially η is well defined and onto homomorphism. Now we show that η is one-one. Let $x + Soc(H_n(K)) + Soc(H_{n+1}(M)) \in \operatorname{Ker} \eta$, then $x \in Soc(H_{n+1}(M)) + K$, so $x = y + k, y \in Soc(H_{n+1}(M)), k \in K$ and we get $x - y = k \in K \cap Soc(H_n(M))$ but K is h-pure in M; hence $x - y \in Soc(H_n(K))$, which yields $x \in Soc(H_n(K)) + Soc(H_{n+1}(M))$. Therefore, $\operatorname{Ker} \eta = 0$ and we get η to be an isomorphism.

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