ON $h$-PURIFIABLE SUBMODULE OF QTAG-MODULE

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Abstract. Different concepts and decomposition theorems have been done for QTAG-modules by a number of authors. The concept of quasi $h$-pure submodules were introduced and different characterizations were obtained in [5]. The purpose of this paper is to obtain the relation between purifiability of a submodule and quasi $h$-pure submodules. Further we obtained results which shows that purifiability of a submodule is very much dependent on the purifiability of a $h$-pure and $h$-dense submodule of the given submodule.

0. Introduction

S. Singh [9] introduced the concept of QTAG-module and did different decomposition theorems. A module $M_R$ is called QTAG-module if it satisfies the condition: Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. Since the different concepts for QTAG-modules have been introduced by different authors and various results based on those concepts have been obtained. In [5], Mohd. Z. Khan and A. Zubair introduced the concept of quasi $h$-pure submodules and obtained various characterizations and their consequences. In this paper we continue the similar study in term of purifiability of submodules and obtained a characterization. We have also established a necessary and sufficient condition for a submodule to be $h$-purifiable.

1. Preliminaries

Rings considered here are with unity ($1 \neq 0$) and modules are unital QTAG-module. A module in which the lattice of its submodules is totally ordered is called a serial module; in addition if it has finite composition length it is called uniserial module. An element $x \in M$ is called uniform if $xR$ is a non zero uniform (hence uniserial) submodule of $M$. If $x \in M$ is uniform then $e(x) = d(xR)$ (The composition length of $xR$), $H_M(x) = \sup \{d(yR/xR) / x \in yR \}

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and \( y \in M \) is uniform \( x \) and height of \( x \) in \( M \) respectively. For any \( n \geq 0, H_n(M) = \{ x \in M | H_M(x) \geq n \} \). A submodule \( N \) of \( M \) is called \( h \)-pure in \( M \) if \( H_k(N) = N \cap H_k(M) \) for all \( k \geq 0, N \) is \( h \)-neat in \( M \) if \( H_1(N) = N \cap H_1(M) \). The module \( M \) is called \( h \)-divisible if \( H_1(M) = M \). For other basic concepts of QTAG-module one may see [2, 3, 4, 7, 8, 9].

2. Purifiability

Firstly we recall the following:

\textbf{Lemma A ([2])}. If \( A \) and \( B \) are any two uniserial submodules of a QTAG-module \( M \) such that \( A \cap B \neq 0 \) and \( d(A) \leq d(B) \). Then there exists a monomorphism \( \sigma : A \rightarrow B \), which is identity on \( A \cap B \).

\textbf{Definition 2.1 ([8])}. A submodule \( N \) of a QTAG-module \( M \) is called \( h \)-dense if \( M/N \) is \( h \)-divisible.

\textbf{Definition 2.2 ([7])}. A submodule \( N \) of a QTAG-module \( M \) is said to be almost \( h \)-dense in \( M \) if for every \( h \)-pure submodule \( K \) of \( M \) containing \( N \), \( M/K \) is \( h \)-divisible.

Now we restate the following:

\textbf{Theorem 2.3 ([7], Theorem 5)}. A submodule \( N \) of a QTAG-module \( M \) is almost \( h \)-dense in \( M \) if and only if \( N + H_n(M) \supseteq \text{Soc}(H_{n-1}(M)) \) for all \( n \geq 1 \).

Now before defining the \( h \)-purifiable submodule, we would like to adopt the following notations and results from [5].

\textbf{Notation 2.4 ([5])}. For any non-negative integer \( t \) and for a submodule \( N \) of a QTAG-module \( M \), we denote by \( N^t(M) \) the submodule \( (N + H_{t+1}(M)) \cap \text{Soc}(H_t(M)) \), by \( N_t(M) \) the submodule \( (N \cap \text{Soc}(H_t(M))) + \text{Soc}(H_{t+1}(M)) \) and by \( Q_t(M, N) = N^t(M)/N_t(M) \).

It is trivial to see that

\[ N^t(M) = (N + H_{t+1}(M)) \cap \text{Soc}(H_t(M)) = \text{Soc}(N \cap H_t(M) + H_{t+1}(M)) \]

and

\[ N_t(M) = (N \cap \text{Soc}(H_t(M))) + \text{Soc}(H_{t+1}(M)) = (\text{Soc}(N))^t(M) \]
Theorem 2.5 ([5], Theorem 4.2). If $N$ and $K$ are submodules of QTAG-module $M$ such that $N \subseteq K$ and $K$ is $h$-pure in $M$, then the module $Q_n(M, N)$ and $Q_n(K, N)$ are isomorphic, for all $n$.

Theorem 2.6 ([5], Theorem 4.3). If $N$ is $h$-neat submodule of $M$, then $N$ is $h$-pure in $M$ if and only if $Q_n(M, N) = 0$ for all $n \in Z^+$.

Now we define $h$-purifiable submodule:

Definition 2.7. A submodule $N$ of a QTAG-module $M$ is called $h$-purifiable in $M$ if there exists a submodule $K$ of $M$ minimal among the $h$-pure submodules of $M$ containing $N$.

Such $K$ is called $h$-pure hull of $N$ in $M$.

Now we restate the following:

Theorem 2.8 ([6]). A submodule $N$ of a QTAG-module $M$ is $h$-purifiable in $M$ if and only if there exists a $h$-pure submodule $K$ of $M$ such that $Soc(H_n(K)) \subseteq N \subseteq K$ for some $n \in Z^+$.

Proposition 2.9. If $N$ is a $h$-purifiable submodule of a QTAG-module $M$; then there exists $m \in Z^+$ such that $Q_n(M, N) = 0$ for all $n \geq m$.

Proof. If $N$ itself is an $h$-pure submodule, then by [5, Theorem 4.7], $Q_n(M, N) = 0$ for all $n \geq 0$. Now appealing Theorem 2.8, we get an $h$-pure submodule $K$ of $M$ and $m \in Z^+$ such that $Soc(H_m(K)) \subseteq N \subseteq K$. Now for $n \geq m$ it is trivial to see that $N^n(K) = (N + H_{n+1}(K)) \cap Soc(H_n(K)) = Soc(H_n(K)) = N_n(K)$. Hence, $Q_n(K, N) = 0$ for all $n \geq m$. Therefore from Theorem 2.5, we get $Q_n(M, N) = 0$ for all $n \geq m$.

Now we generalize [1, Theorem 66.3] and is of very interesting nature.

Theorem 2.10. If $M$ is a QTAG-module then every $h$-dense subsocle of $M$ supports an $h$-pure and $h$-dense submodule.

Proof. Let $S$ be a subsocle of $M$ and $S$ be $h$-dense; then $Soc(M) = S + Soc(H_k(M))$ for all $k \in Z^+$. Let $N$ be maximal with the property $Soc(N) = S$. Firstly we show that $N$ is $h$-neat submodule of $M$. Let $x$ be a uniform element in $N \cap H_1(M)$, then for a uniform element $y \in M$, we have $d(yR/xR) = 1$. If $y \in N$, then $x \in H_1(N)$. Let $y \notin N$ then $S \subseteq Soc(N + yR)$. Hence, there exists a uniform element $z \in Soc(N + xR)$ such that $z \notin S$ and $z = u + yr$ where $u \in N$ and $r \in R$. Trivially $yrR = yR$, hence without any loss of generality we can assume $z = u + y$. Define $\eta : yR \rightarrow uR$ such that $\eta(yr) = ur$. Let $yr = 0$, then $zr = ur$. If $zrR = zR$ then $z \in S$, a contradiction, therefore $zr = 0$ and we get $ur = 0$, consequently $\eta$ is a well defined epimorphism. Therefore, $uR$ is a uniform submodule. Since $u+y \in Soc(M)$, $H_1(uR) =$
$H_1(yR)$, but $xR$ is a maximal submodule of $M$; hence $H_1(yR) = xR$ and we get $x \in H_1(N)$. Thus, $N \cap H_1(M) = H_1(N)$. Now suppose $N \cap H_n(M) = H_n(N)$ and let $x$ be a uniform element in $N \cap H_{n+1}(M)$; then $d(yR/xR) = 1$ for some uniform element $y \in H_n(M)$. Since $N$ is $h$-neat in $M$, there is a uniform element $y' \in N$ such that $d(y'xR/xR) = 1$. Hence by Lemma A, there is an isomorphism $\sigma : yR \to y'R$ which is identity on $xR$. The map $\eta : yR \to (y - y')R$ where $\sigma(y) = y'$ is an epimorphism with $xR \subseteq \ker \eta$. Hence, $e(y - y') \leq 1$ and we get $y - y' \in \soc(M) = S + \soc(H_n(M))$. Therefore, $y - y' = s + t$ for some $s \in S, t \in H_n(M)$. Consequently, $y - t = y' + s \in N \cap H_n(M) = H_n(N)$. Since $y - y' = s \in \soc(M), H_1(yR) = H_1((y' + s)R) \subseteq H_{n+1}(N)$. Hence, $x \in H_{n+1}(N)$. Therefore, $N$ is $h$-dense submodule of $M$.

Now let $\tilde{x} \in \soc(M/N) = (\soc(M) + N)/N$ be a uniform element; then by [Lemma 3.9, 9] there exists a uniform element $x' \in M$ such that $\tilde{x} = \tilde{x}'$ and $e(x') = 1$. Since $\soc(M) = S + \soc(H_k(M))$ for all $k$, we get $\tilde{x} \in H_k(M/N)$ for every $k$. Hence, $\tilde{x} \in \cap_{k=1}^{\infty} H_k(M/N)$

**Observation:** Using the notations used earlier, the $h$-purity can be established as: Since $\soc(M) = \soc(N) + \soc(H_k(M))$ for all $k \in Z^+$, it is easy to see that $N^n(M) = N_n(M)$ and $Q_n(M, N) = 0$ for all $n \in Z^+$. Since $N$ is $h$-neat, therefore by [5, Theorem 4.7], $N$ is $h$-pure in $M$.

**Theorem 2.11.** If $N$ is almost $h$-dense submodule of a QTAG-module $M$. Then $N$ is $h$-purifiable in $M$ if and only if there exists $m \in Z^+$ such that $Q_n(M, N) = 0$ for all $n \geq m$.

**Proof.** Let $N$ be $h$-purifiable then by Theorem 2.9, we get $Q_n(M, N) = 0$ for all $n \geq m$. Conversely, suppose that $Q_n(M, N) = 0$ for all $n \geq m$ and $N$ is almost $h$-dense in $M$. Then $N^n(M) = N_n(M) = \soc(N \cap H_n(M)) + \soc(H_{n+1}(M))$. Since $N$ is almost $h$-dense in $M$, therefore by Theorem 2.3, we get $\soc(H_n(M)) = \soc(N \cap H_n(M)) + \soc(H_{n+1}(M))$ for all $n \geq m$. Therefore, $\soc(N \cap H_m(M))$ is $h$-dense subsocle of $H_m(M)$. Now appealing to Theorem 2.10, we can find an $h$-pure submodule $K$ of $H_m(M)$ such that $\soc(K) \subseteq N \cap H_m(M) \subseteq K$. It is easy to see that $H_m(M)/K$ is $h$-divisible submodule of $M/K$ and $H_m(M)/K \cap (N + K)/K = 0$. Hence there exists a submodule $T/K$ such that $(N + K)/K \subseteq T/K$ and $M/K = H_m(M)/K \oplus T/K$. Now by [Proposition 2.5, 4], $T$ is $h$-pure submodule of $M$. Trivially $T \cap H_m(M) = K$, but $T \cap H_m(M) = H_m(T)$; so $H_m(T) = K$. Hence, $\soc(H_m(K)) \subseteq \soc(K) \subseteq N$. Hence by Theorem 2.8, we get $N$ to be $h$-purifiable.

### 3. Role of $h$-pure and $h$-dense submodules

In this section we show that $h$-purifiability of a submodule depends upon the $h$-purifiability of an $h$-pure and $h$-dense submodule of the given submodule.
Firstly we prove the following results for obtaining a necessary and sufficient condition for $h$-purifiability.

**Theorem 3.1.** *If $B$ is an $h$-pure and $h$-dense submodule of a submodule $K$ of a QT AG-module $M$, then $Q_n(M, N) = Q_n(M, B)$ for all $n \in \mathbb{Z}^+$.***

**Proof.** Since $B$ is $h$-dense in $K$, then we have $K = B + H_{n+1}(K)$ for all $n \geq 0$ and hence, $K + H_{n+1}(M) = B + H_{n+1}(M)$. Therefore, $K^n(M) = B^n(M)$ for all $n \geq 0$.

Further, $K_n(M) = (\text{Soc}(K))^n(M) = (\text{Soc}(K) + H_{n+1}(M)) \cap \text{Soc}(H_n(M))$. Now appealing to [3, Prop.6], we get

$$K_n(M) = \left(\text{Soc}(B) + \text{Soc}(H_{n+1}(K)) + H_{n+1}(M)\right) \cap \text{Soc}(H_n(M))$$

$$= \left(\text{Soc}(B) + H_{n+1}(M)\right) \cap \text{Soc}(H_n(M))$$

$$= B_n(M)$$

Hence, $Q_n(M, K) = Q_n(M, B)$.

**Proposition 3.2.** *If $B$ is an $h$-pure and $h$-dense submodule of a submodule $K$ of a QT AG-module $M$. If $K$ is $h$-purifiable in $M$, then $B$ is $h$-purifiable in $M$.***

**Proof.** Let $T$ be a $h$-pure hull of $K$ in $M$. Since $B$ is $h$-dense in $K$ we get, $K/B$ is $h$-divisible, so $T/B = K/B \oplus L/B$. Appealing to [Proposition 2.5, 4] we get, $L$ to be a $h$-pure submodule of $T$ and hence $L$ is $h$-pure in $M$. Let $N$ be a $h$-pure submodule of $M$ such that $B \subseteq N \subseteq L$. Then we claim that $K + N$ is a $h$-pure submodule of $M$. Since $K = B + H_n(K)$, we have $K + N = H_n(K) + N$. Therefore,

$$(K + N) \cap H_n(M) = (H_n(K) + N) \cap H_n(M)$$

$$= H_n(K) + (N \cap H_n(M))$$

$$= H_n(K) + H_n(N)$$

$$= H_n(K + N)$$

for all $n \geq 0$.

Since $T$ is a $h$-pure hull of $K$ in $M$, we have $K + N = T$ and

$$L = (K + N) \cap L = N + (K \cap L) = N + B = N$$

Therefore, $L$ is a $h$-pure hull of $B$ in $M$.

**Proposition 3.3.** *If $B$ is a $h$-pure and $h$-dense submodule of a submodule $K$ of a QT AG-module $M$ and if $N$ be a $h$-pure hull of $B$ in $M$ and $\text{Soc}(N) = \text{Soc}(B)$, then $K + N$ is a $h$-pure hull of $K$ in $M$.***
Proof. Since \( K/B \) is \( h \)-divisible, we have \( K = B + H_n(K) \). Now \( K + N = B + H_n(K) + N = N + H_n(K) \) and hence \((K + N) \cap H_n(M) = (N + H_n(K)) \cap H_n(M) = H_n(K) + N \cap H_n(M) = H_n(K) + H_n(N) = H_n(K + N) \) for all \( n \geq 0 \). Therefore, \( K + N \) is \( h \)-pure submodule of \( M \). Since \( \text{Soc}(N) = \text{Soc}(B) \), so \( \text{Soc}(K \cap N) = \text{Soc}(B) \), so \( N \cap K \) is an essential extension of \( B \) in \( K \). Since \( h \)-pure submodules have no proper essential extensions, therefore we get, \( K \cap N = B \). Now we show that \( \text{Soc}(K + N) = \text{Soc}(K) \), which will yield that \( N + K \) is \( h \)-pure hull of \( K \) in \( M \). Using [Lemma 1, 2] we can proceed as: If \( x \in \text{Soc}(K + N) \) then \( H_1(xR) = 0 \) and \( x = k + t \) where \( k \in K, t \in N \), then \( H_1(tR) = H_1(kR) \subseteq N \cap K = B \cap H_1(K) = H_1(B) \). Hence, \( H_1(tR) = H_1(kR) = H_1(bR) \) for \( b \in B \). Hence, \( k - b \in \text{Soc}(K) \) and \( t + b, t - b \in \text{Soc}(N) = \text{Soc}(B) \). Hence \( x = k - b + b + t \in \text{Soc}(K) \) and we get \( \text{Soc}(K + N) = \text{Soc}(K) \).

**Proposition 3.4.** If \( K \) is a \( h \)-pure hull of a submodule \( N \) of a QTAG-module \( M \) such that \( \text{Soc}(K) \neq \text{Soc}(N) \). Then there exists \( m \in \mathbb{Z}^+ \) such that \( Q_m(M, N) \neq 0 \).

**Proof.** From Theorem 2.8 and Theorem 2.3, there exists \( n \in \mathbb{Z}^+ \) such that \( \text{Soc}(H_n(K)) \subseteq N \) and \( \text{Soc}(H_t(K)) \subseteq N + H_{t+1}(K) \) for all \( t \geq 0 \). Since \( \text{Soc}(K) \neq \text{Soc}(N) \), the smallest \( n \) such that \( \text{Soc}(H_n(K)) \subseteq N \), we have \( n \neq 0 \).

Now taking \( n = m - 1, N^m(K) = \text{Soc}(H_m(K)) \) while \( N_m(K) \subseteq N \). Therefore, \( N^m(K) \neq N_m(K) \) but by [5, Theorem 2.1], \( Q_m(M, N) \equiv Q_m(M, K) \neq 0 \). Hence, \( Q_m(M, N) \neq 0 \).

**Proposition 3.5.** Let \( N \) be a submodule of a QTAG-module \( M \). If \( N \) is \( h \)-purifiable in \( M \), then \( N \cap H_n(M) \) is \( h \)-purifiable in \( H_n(M) \) for all \( n \geq 0 \). Conversely, if \( N \cap H_n(M) \) is \( h \)-purifiable in \( H_n(M) \) for some \( n \geq 1 \), then \( N \) is \( h \)-purifiable in \( M \).

**Proof.** Let \( K \) be \( h \)-pure hull of \( N \) in \( M \), then trivially \( H_n(K) \) is \( h \)-pure submodule \( H_n(M) \) for all \( n \in \mathbb{Z}^+ \). Also \( H_n(K) = K \cap H_n(M) \supseteq N \cap H_n(M) \).

Now we claim that \( H_n(K) \) is \( h \)-pure hull of \( N \cap H_n(M) \) in \( H_n(M) \). Let \( T \) be \( h \)-pure submodule of \( H_n(M) \) such that \( H_n(K) \supseteq T \supseteq N \cap H_n(M) \). Trivially \( N \cap H_n(K) \subseteq N \cap H_n(M) \) and \( N \cap H_n(K) \supseteq T \cap N \supseteq N \cap H_n(M) \); consequently \( H_n(K) \supseteq T \supseteq N \cap H_n(M) = N \cap H_n(K) \). Now appealing to [Theorem 4.12, 5], we can extend \( N + T \) to an \( h \)-pure submodule \( D \) of \( K \) such that \( D \cap H_n(K) = T \) (we can note that \((N + T) \cap H_n(K) = T + N \cap H_n(K) = T \)). Thus, \( D = K \) and we get \( H_n(K) = T \). Hence, \( H_n(K) \) is \( h \)-pure hull of \( N \cap H_n(M) \) in \( H_n(M) \). Conversely, let \( N \cap H_n(M) \) be \( h \)-purifiable in \( H_n(M) \) and \( T \) be \( h \)-pure hull of \( N \cap H_n(M) \) in \( H_n(M) \). Then as done above \((N + T) \cap H_n(M) = T \) and \( N + T \) can be extended to an \( h \)-pure submodule \( K \) of \( M \) such that \( K \cap H_n(M) = T \). Clearly \( T = H_n(K) \). Appealing to Theorem 2.8 there exists \( m \in \mathbb{Z}^+ \) such that \( \text{Soc}(H_m(T)) \subseteq H_n(M) \); so \( \text{Soc}(H_m+1(K)) \subseteq N \subseteq K \). Hence by Theorem 2.8, \( N \) is \( h \)-purifiable in \( M \).

**Theorem 3.6.** If \( N \) is a submodule of a QTAG-module \( M \). Then \( N \) is \( h \)-purifiable if and only if all basic submodules of \( N \) are \( h \)-purifiable.
Proof. Let all basic submodules of $N$ be $h$-purifiable. Then by Theorem 3.1 and Theorem 2.11, there exists $m \in \mathbb{Z}^+$ such that $Q_n(M, N) = 0$ for all $n \geq m$. Hence $Q_n(H_m(M), N \cap H_m(M)) = 0$ for all $n \geq 0$. Let $B$ be a basic submodule of $N \cap H_n(M)$; then $N/B = (N \cap H_n(M))/B \oplus T/B$ and we get $T$ to be $h$-pure in $N$ (see [Proposition 2.5, 4]) also $T/B \cong N/(N \cap H_m(M)) \cong (N + H_m(M))/H_m(M)$ is trivially bounded. Hence, $T$ is also a direct sum of uniserial modules and we get $T$ to be a basic submodule of $N$. As given, $T$ is $h$-purifiable in $M$, therefore $T \cap H_m(M) = B$ is $h$-purifiable in $H_m(M)$ by Proposition 3.5; consequently $B$ is $h$-purifiable basic submodule of $N \cap H_m(M)$ in $H_m(M)$, and $Q_n(H_m(M), B) = 0$ for all $n \geq 0$. Now let $L$ be a $h$-pure hull of $B$ in $H_m(M)$, then $Q_n(L, B) = 0$ for all $n \geq 0$, and by Proposition 3.4, $Soc(L) = Soc(B)$. Hence by Proposition 3.3, $N \cap H_m(M)$ is $h$-purifiable in $H_m(M)$ and so by Proposition 3.5, $N$ is $h$-purifiable in $M$. The converse follows from Theorem 3.2.

Lastly we prove the following result which is of particular interest.

Theorem 3.7. If $N$ is almost $h$-dense submodule of a QTAG-module $M$ and $K$ is $h$-pure hull of $Soc(N)$. Then $Q_n(M, N) \cong (Soc(H_n(M)) + K)/(Soc(H_{n+1}(M)) + K)$ for all $n \in \mathbb{Z}^+$.

Proof. As $N$ is almost $h$-dense in $M$, then appealing to Theorem 2.3, we have $N^n(M) = Soc(H_n(M))$ Since $K$ is $h$-pure hull of $Soc(N)$ in $M$, $Soc(K) = Soc(N)$. Therefore, $N_n(M) = Soc(N \cap H_n(M)) + Soc(H_{n+1}(M)) = Soc(H_n(K)) + Soc(H_{n+1}(M))$. So we get $Q_n(M, N) = Soc(H_n(M))/(Soc(H_n(K)) + Soc(H_{n+1}(M)))$. Now we define a map $\eta : Q_n(M, N) \longrightarrow (Soc(H_n(K)) + K)/(Soc(H_{n+1}(M)) + K)$ given as $\eta(x + Soc(H_n(K)) + Soc(H_{n+1}(M))) = x + Soc(H_{n+1}(M)) + K$. Then trivially $\eta$ is well defined and onto homomorphism. Now we show that $\eta$ is one-one. Let $x + Soc(H_n(K)) + Soc(H_{n+1}(M)) \in Ker \eta$, then $x \in Soc(H_{n+1}(M)) + K$, so $x = y + k$, $y \in Soc(H_{n+1}(M))$, $k \in K$ and we get $x - y = k \in K \cap Soc(H_n(M))$ but $K$ is $h$-pure in $M$; hence $x - y \in Soc(H_n(K))$, which yields $x \in Soc(H_n(K)) + Soc(H_{n+1}(M))$. Therefore, Ker $\eta = 0$ and we get $\eta$ to be an isomorphism.

References


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