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LYAPUNOV-TYPE INEQUALITY FOR THIRD-ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we give a proof of a Lyapunov-type inequality for third-order half-linear differential equations. Then some applications, e.g. the distance between consecutive zeros of a solution, are studied with the help of the inequality.

1. Introduction

In this paper, we generalize the Lyapunov inequality for linear third-order differential equation

$$y''' + p(t)\, y = 0 \tag{1}$$

for a class of half-linear differential equations of the third-order. It is well known [6] that if $p \in C[a,b]$ and $x(t)$ is nonzero solution of (1) s.t. $x(a) = x(b) = 0$ $(a < b)$ and there exist $d \in [a,b]$ s.t. $y''(d) = 0$, then the following inequality holds:

$$\int_a^b |p(t)|\,\mathrm{d}s > \frac{4}{(b-a)^2}. \tag{2}$$

Such result has found many practical uses in problems as oscillation theory or eigenvalue problems (spectral properties of differential equations). There are several generalizations in the literature. For higher-order linear differential equations see e.g. [1, 8] and for certain nonlinear higher-order differential equations see [5]. Development of theory of differential equations together with practical problems bring also delayed type of equations. If one is interested along this line see [7], where authors handle with the third-order delay differential equations. We study here a special type of nonlinear differential equations, of which the solution space possesses homogeneity property but lacks for additivity property. Second-order half-linear differential equations have been widely studied in recent years and there is a nice

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overview in the monograph [2]. Less literature exists, which deals with such equations of higher-order (especially odd-order differential equations), but one can see for example [4].

2. Main results

We are concerned with the third-order half-linear differential equation

$$\left(\frac{1}{r_2(t)}\Phi_{\alpha_2}\left[\left(\frac{1}{r_1(t)}\Phi_{\alpha_1}[y']\right)'\right]\right)' + q(t)\Phi_\beta[y] = 0, \tag{E}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $q \in C([a,b],\mathbb{R})$, and $\Phi_\alpha[x] := |x|^{\alpha-1}x$, $\alpha > 0$, known as signed-power function. Moreover we assume that $(r_k)^{-1} \in C^{3-k}([a,b],(0,\infty))$, $k = 1,2$, and to preserve mentioned homogeneity property we also demand that $\beta = \alpha_1\alpha_2$. Equation (E) can be written by means of quasi-derivatives with respect to the coefficients r_i and functions Φ_{α_i}, $i = 1,2$. We will denote them as follows:

$$\begin{aligned} \mathrm{D}^{(0)}y(t) &= y(t), \\ \mathrm{D}^{(i)}y(t) &= \frac{1}{r_i(t)}\Phi_{\alpha_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(i-1)}y(t)\right] \quad i = 1,2 \\ \mathrm{D}^{(3)}y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(2)}y(t). \end{aligned} \tag{4}$$

A solution of (E) is said to be oscillatory (nonoscillatory) if it has (has not) a sequence of zeros converging to infinity. Equation (E) is oscillatory if all its solutions are oscillatory and nonoscillatory otherwise. If a solution of (E) has two consecutive zeros $a < b$, then there can two cases occur. Either there exist a $d \in [a,b]$ s.t. $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}y(d) = 0$ or $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}y(t) \neq 0$ for $t \in [a,b]$. The first case illustrates the following assertion.

Theorem 2.1. *If $y(t)$ is a nonzero solution of (E) satisfying $y(a) = y(b) = 0$ and there exist a $d \in [a,b]$ s.t. $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}y(d) = 0$. Then*

$$2\left(\int_a^b |q(t)|\,\mathrm{d}t\right)^{\frac{1}{\beta}} > \min_{c\in[a,b]} h(c), \tag{5}$$

where $h(c) = \dfrac{1}{\int_a^c r_1^{\frac{1}{\alpha_1}}\,\mathrm{d}t\left(\int_a^c r_2(t)^{\frac{1}{\alpha_2}}\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}}} + \dfrac{1}{\int_c^b r_1^{\frac{1}{\alpha_1}}\,\mathrm{d}t\left(\int_c^b r_2(t)^{\frac{1}{\alpha_2}}\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}}}.$

Proof. We first define functions y_k, $k = 0,1,2$, as follows:

$$\begin{aligned} y_0 &:= \mathrm{D}^{(0)}y, \\ y_i &:= \mathrm{D}^{(i)}y(t) = \frac{1}{r_i(t)}\Phi_{\alpha_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}y_{i-1}\right], \qquad i = 1,2 \end{aligned} \tag{6}$$

Equation (E) is then equivalent to the following differential system:

$$\begin{aligned} y_i' &= r_{i+1}(t)^{\frac{1}{\alpha_{i+1}}} \Phi^{-1}_{\alpha_{i+1}}\left[y_{i+1}\right], \qquad && i = 0,1 \\ y_2' &= -q(t)\Phi_\beta[y_0]. && \end{aligned} \tag{7}$$

Condition $y_0(a) = y_0(b) = 0$ gives us existence of $c \in (a,b)$ s.t. $y_0'(c) = 0$ and $|y_0(c)| = \max_{t\in[a,b]} |y_0(t)|$. It follows from the latter that $y_1(c) = 0$. By integrating the first equation of the system (7) from a to c we obtain

$$y_0(c) = \int_a^c r_1^{\frac{1}{\alpha_1}}(t)\,\Phi^{-1}_{\alpha_1}[y_1(t)]\,\mathrm{d}t,$$

which implies

$$|y_0(c)| \le \int_a^c r_1^{\frac{1}{\alpha_1}}(t)\,|y_1(t)|^{\frac{1}{\alpha_1}}\,\mathrm{d}t. \tag{8}$$

Now let t be in $[a,c]$. From the fact that $y_1(c) = 0$ and $y_1(t) = \int_c^t y_1'(s)\,\mathrm{d}s$ we get

$$|y_1(t)| \le \int_a^c r_2^{\frac{1}{\alpha_2}}(t)\,|y_2(t)|^{\frac{1}{\alpha_2}}\,\mathrm{d}t. \tag{9}$$

Further, from the second condition of the assumptions and relations (6) or (7), we know that $y_2(d) = 0$, which implies $y_2(t) = -\int_d^t q(s)\,\Phi_\beta[y_0(s)]\,\mathrm{d}s$ for $t \in [a,c]$. Moreover we have

$$|y_2(t)| < |y_0(c)|^\beta \int_a^b |q(t)|\,\mathrm{d}t. \tag{10}$$

Combining inequalities (8)-(10), we get

$$|y_0(c)| < |y_0(c)| \int_a^c r_1^{\frac{1}{\alpha_1}}(t)\,\mathrm{d}t \left(\int_a^c r_2^{\frac{1}{\alpha_2}}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}} \left(\int_a^b |q(t)|\,\mathrm{d}t\right)^{\frac{1}{\beta}},$$

which implies

$$1 < \int_a^c r_1^{\frac{1}{\alpha_1}}(t)\,\mathrm{d}t \left(\int_a^c r_2^{\frac{1}{\alpha_2}}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}} \left(\int_a^b |q(t)|\,\mathrm{d}t\right)^{\frac{1}{\beta}}. \tag{11}$$

Analogously, we can get

$$1 < \int_c^b r_1^{\frac{1}{\alpha_1}}(t)\,\mathrm{d}t \left(\int_c^b r_2^{\frac{1}{\alpha_2}}(t)\,\mathrm{d}t\right)^{\frac{1}{\alpha_1}} \left(\int_a^b |q(t)|\,\mathrm{d}t\right)^{\frac{1}{\beta}}. \tag{12}$$

But (11) and (12) together imply (5). Moreover, it is obvious that h takes its minimum in (a,b), since it is continuous there and $\lim_{c\to a^+} h(c) = \lim_{c\to b^-} h(c) = \infty$. □

In the case that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\,y(t) \neq 0$ for $t \in [a,b]$, we consider three consecutive zeros of $y(t)$. We give only sketch of the proof as it is almost copy of the previous one.

Theorem 2.2. *If $y(t)$ is a nonzero solution of* (E) *satisfying* $y(a) = y(d) = y(b) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\, y(t) \neq 0$ *for* $t \in [a,d]$ *and* $y(t) \neq 0$ *for* $t \in (a,d) \cup (d,b)$. *Then inequality* (5) *holds.*

Proof. Conditions $y_0(a) = y_0(d) = y_0(b) = 0$ give us existence of $c_1 \in (a,d)$, $c_2 \in (d,b)$, s.t. $y_1(c_1) = y_1(c_2) = 0$ and further application of Rolle's theorem gives us existence of $e \in (c_1, c_2)$, s.t. $y_2(e) = 0$. Denoting by $c \in (a,d) \cup (d,b)$ a point where $\max_{t\in[a,b]} |y_0(t)| = |y_0(c)|$ and using previous procedure it can be proved that inequality (5) holds. Notice, that there can not be problem with continuity of h on (a,b). ☐

Remark 2.1. Put $r_1(t) = r_2(t) = 1$. Since h attains minimum at $c = \frac{a+b}{2}$, then (5) reduces to

$$\int_a^b |q(t)|\,\mathrm{d}t > \left(\frac{2}{b-a}\right)^{\alpha_2(\alpha_1+1)}.$$

Notice that in case $\alpha_1 = \alpha_2 = 1$ this inequality reduces to (2), which appears in the classical result.

Remark 2.2. Put $r_1(t) = r_2(t) = r(t)$ and $\alpha_1 = \alpha_2 = \alpha$ then (5) reduces to

$$\int_a^b |q(t)|\,\mathrm{d}t > \left(\frac{2}{\int_a^b r(t)^{\frac{1}{\alpha}}\,\mathrm{d}t}\right)^{\alpha(\alpha+1)},$$

since $\int_a^c r(t)^{\frac{1}{\alpha}}\,\mathrm{d}t = \int_c^b r(t)^{\frac{1}{\alpha}}\,\mathrm{d}t = \frac{1}{2}\int_a^b r(t)^{\frac{1}{\alpha}}\,\mathrm{d}t$ must hold for c in order to minimize h on (a,b) $\left(c = R^{-1}\left(\frac{R(b)+R(a)}{2}\right)\right.$, where R is antiderivative of $\left. r^{\frac{1}{\alpha}}\right)$.

3. Applications

Further we introduce some applications of the previous results for reduced equation (E).

Theorem 3.1. *Let $y(t)$ be an oscillatory solution of the reduced* ($r_1(t) = r_2(t) = 1$) *equation* (E) *with increasing sequence of zeros* $\{t_k\}_{k=1}^{\infty}$ *and* $q \in L^{\mu}([0,\infty],\mathbb{R})$, $\mu \in [1,\infty)$. *Then distances between consecutive zeros* $\{t_{k+1} - t_k\}$ *or* $\{t_{k+2} - t_k\}$ *goes to infinity.*

Proof. In a proof by contradiction we suppose that, in the case that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\, y(t) = 0$ for some $t \in [t_k, t_{k+1}]$ for every large k, is not true that $\{t_{k+1} - t_k\} \to \infty$. Hence, there exist a subsequence $\{t_{k_n}\}_{n=1}^{\infty}$ s.t. $(t_{k_{n+1}} - t_{k_n}) < K$ for every n, $(K > 0)$. Let $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\, y(c_{k_n}) = 0$ for $c_{k_n} \in [t_{k_n}, t_{k_{n+1}}]$ and $\max_{t\in[t_{k_n}, t_{k_{n+1}}]} |y(t)| = |y(d_{k_n})|$ where $d_{k_n} \in (t_{k_n}, t_{k_{n+1}})$. Without loss of generality we can assume that $c_{k_n} < d_{k_n}$. Then it follows that

$$\int_{t_{k_n}}^{t_{k_{n+1}}} |q(t)|\,\mathrm{d}t > \left(\frac{2}{d_{k_n} - t_{k_n}}\right)^{\beta+\alpha_2}.$$

From integrability of q we have

$$\int_{t_{k_n}}^{\infty} |q(t)|^{\mu}\,\mathrm{d}t < \left(\frac{2^{\beta+\alpha_2}}{K^{\beta+\alpha_2+\frac{1}{\tilde{\mu}}}}\right)^{\mu},$$

for sufficiently large n and $\frac{1}{\mu}+\frac{1}{\tilde{\mu}}=1$. Therefore using Hölder inequality we obtain

$$1 < \left(\frac{d_{k_n}-t_{k_n}}{2}\right)^{\beta+\alpha_2}\int_{t_{k_n}}^{t_{k_{n+1}}} |q(t)|\,\mathrm{d}t \le \frac{(t_{k_{n+1}}-t_{k_n})^{\beta+\alpha_2+\frac{1}{\tilde{\mu}}}}{2^{\beta+\alpha_2}}\left(\int_{t_{k_n}}^{t_{k_{n+1}}} |q(t)|^{\mu}\,\mathrm{d}t\right)^{\frac{1}{\mu}} \le$$

$$\le \frac{(t_{k_{n+1}}-t_{k_n})^{\beta+\alpha_2+\frac{1}{\tilde{\mu}}}}{2^{\beta+\alpha_2}}\left(\int_{t_{k_n}}^{\infty} |q(t)|^{\mu}\,\mathrm{d}t\right)^{\frac{1}{\mu}} < \frac{K^{\beta+\alpha_2+\frac{1}{\tilde{\mu}}}}{2^{\beta+\alpha_2}}\left(\frac{2^{\beta+\alpha_2}}{K^{\beta+\alpha_2+\frac{1}{\tilde{\mu}}}}\right) = 1,$$

a contradiction.

Now suppose that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\,y(t) \neq 0$ for $t \in [t_k, t_{k+1}]$ (for some large k). In this case we consider three consecutive zeros $t_k < t_{k+1} < t_{k+2}$. Suppose that there exist subsequence $\{t_{k_n}\}_{n=1}^{\infty}$ s.t. $(t_{k_{n+1}}-t_{k_n}) < M$ for every n, $(M>0)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\,y(t) \neq 0$ for $t \in [t_{k_n}, t_{k_{n+1}}]$. Since $y(t_{k_{n+2}})=0$, there exists a $c_{k_n} \in [t_{k_{n+1}}, t_{k_{n+2}}]$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\,y(c_{k_n})=0$. Now set $\max_{t\in[t_{k_n},t_{k_{n+2}}]} |y(t)| = |y(d_{k_n})|$ where $d_{k_n} \in (t_{k_n}, t_{k_{n+1}}) \cup (t_{k_{n+1}}, t_{k_{n+2}})$. If $d_{k_n} \in (t_{k_{n+1}}, t_{k_{n+2}})$ then we can proceed as in the previous part of the proof. If $d_{k_n} \in (t_{k_n}, t_{k_{n+1}})$, then it follows that

$$\int_{t_{k_n}}^{t_{k_{n+2}}} |q(t)|\,\mathrm{d}t > \left(\frac{2}{d_{k_n}-t_{k_n}}\right)^{\beta+\alpha_2}.$$

Therefore using Hölder inequality we obtain a contradiction as in the first part of the proof. □

The following theorems give us an estimation (upper bound) of the number of zeros of an oscillatory solution of reduced equation (E) on bounded interval $[0,T]$.

Theorem 3.2. *If $y(t)$ is an oscillatory solution of reduced $(r_1(t)=r_2(t)=1)$ equation (E) with zeros $0<t_1<t_2<\cdots<t_N\le T$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}\,y(e_k)=0$ for some $e_k \in [t_k, t_{k+1}]$, $k=1,2,\dots,N-1$. Moreover let $\beta+\alpha_2\ge 1$, then*

$$T^{\beta+\alpha_2}\int_0^T |q(t)|\,\mathrm{d}t > 2^{\beta+\alpha_2}(N-1)^{\beta+\alpha_2+1}.$$

Proof. We know that $\int_{t_k}^{t_{k+1}} |q(t)|\,\mathrm{d}t > \left(\frac{2}{t_{k+1}-t_k}\right)^{\beta+\alpha_2}$, $k=1,\dots,N-1$. Thus,

$$\int_0^T |q(t)|\,\mathrm{d}t \ge \int_{t_1}^{t_N} |q(t)|\,\mathrm{d}t > \sum_{k=1}^{N-1}\left(\frac{2}{t_{k+1}-t_k}\right)^{\beta+\alpha_2}.$$

Now, using known inequality for the power mean with exponent $\beta+\alpha_2$ and arithmetic mean and inequality $\frac{1}{n}\sum_{i=1}^{n} A_i \ge \left(\prod_{i=1}^{n} A_i\right)^{\frac{1}{n}} \ge \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{A_i}\right)^{-1}$, $A_i>0,\ 1\le i\le n$, we obtain

$$\sum_{k=1}^{N-1}\left(\frac{1}{t_{k+1}-t_k}\right)^{\beta+\alpha_2} \ge (N-1)\left(\frac{1}{N-1}\sum_{k=1}^{N-1}\frac{1}{t_{k+1}-t_k}\right)^{\beta+\alpha_2} \ge (N-1)^{\beta+\alpha_2+1}\left(\sum_{k=1}^{N-1}(t_{k+1}-t_k)\right)^{-\beta-\alpha_2}$$
$$= (N-1)^{\beta+\alpha_2+1}(t_N-t_1)^{-\beta-\alpha_2} > \frac{(N-1)^{\beta+\alpha_2+1}}{T^{\beta+\alpha_2}}.$$

This completes the proof. □

Theorem 3.3. *If $y(t)$ is an oscillatory solution of reduced $(r_1(t)=r_2(t)=1)$ equation (E) with zeros $0<t_1<t_2<\cdots<t_{2N+1}\le T$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}^{(1)}y(t)\ne 0$ for $t\in[t_{2k-1},t_{2k}]$, $k=1,2,\ldots,N$. Moreover let $\beta+\alpha_2\ge 1$, then*

$$T^{\beta+\alpha_2}\int_0^T |q(t)|\,\mathrm{d}t > 2^{\beta+\alpha_2}N^{\beta+\alpha_2+1}.$$

The proof can be omitted as one can proceed similarly as in Theorem 3.2. We left the case $\beta+\alpha_2\in(0,1)$ as an open problem.

Example 3.1. For simplicity we consider exponents α_1, α_2 to be the quotients of two odd numbers. Moreover, let the coefficients of the quasi-derivatives be identically constant. We study the following generalized Euler's differential equation on $\mathbb{R}^+$

$$([([y']^{\alpha_1})']^{\alpha_2})' + \frac{\gamma}{(t+1)^{\beta+\alpha_2+1}}\,y^{\beta} = 0. \tag{13}$$

We can proceed using the analogy with the linear Euler differential equation. If we denote as $D=(\alpha_1+\alpha_2)^2+4\beta(\beta+\alpha_2)$, then the roots of an algebraic (indical) equation corresponding to a solution t^{λ} are

$$\lambda_{\pm} = \frac{\alpha_1+2\beta(1+\alpha_1)+\alpha_2\pm\sqrt{D}}{2\alpha_1(1+\beta+\alpha_2)}.$$

Although it has not been proven yet and it is only a conjecture, see [3], we believe that it can be shown the following. Constants $\gamma_{\pm}=\lambda_{\pm}^{\beta}(\lambda_{\pm}-1)^{\alpha_2}\alpha_1^{\alpha_2}(\beta(\lambda_{\pm}-1)-\alpha_2)$ decide whether (13) is oscillatory or not (notice, that in the linear case $\gamma_{\pm}=2\sqrt{3}/9$). Be more precise, a conjecture states that if $\gamma\in[\gamma_-,\gamma_+]$ then equation (13) is nonoscillatory, otherwise it is oscillatory. Thus we can state at least estimate for γ. So, from the previous reflections we have

$$\frac{|\gamma|\,(1-(T+1)^{-\beta-\alpha_2})\,T^{\beta+\alpha_2}}{\beta+\alpha_2} > 2^{\beta+\alpha_2}(N-1)^{\beta+\alpha_2+1}.$$

Example 3.2. Finally we give an application of the obtained result for the following eigenvalue problem

$$\begin{aligned} &\mathrm{D}^{(3)} y \pm \lambda q(t) \Phi_\beta[y] = 0, \\ &y(a) = y(c) = y(b) = 0, \quad a < c < b. \end{aligned} \tag{14}$$

Let the assumptions of Theorem 2.1 be fulfilled, then it follows that

$$|\lambda| > \frac{H^\beta}{2^\beta \int_a^b |q(t)| \,\mathrm{d}t},$$

where $H = \min_{c \in [a,b]} h(c)$ and h is function defined in Theorem 2.1. Especially for the reduced problem ($r_1(t) = r_2(t) = 1$) we obtain

$$|\lambda| > \frac{2^{\beta+\alpha_2}}{(b-a)^{\beta+\alpha_2} \int_a^b |q(t)| \,\mathrm{d}t}.$$

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