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STRONG CONVERGENCE THEOREM FOR TWO ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS WITH ERRORS IN BANACH SPACE

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Abstract. In this paper, we study strong convergence of common fixed points of two asymptotically quasi-nonexpansive mappings and prove that if K is a nonempty closed convex subset of a real Banach space E and let $S, T\colon K \to K$ be two asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}$, $\{v_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \phi$. Suppose $\{x_n\}_{n=1}^{\infty}$ is generated iteratively by $x_1 \in K$, and

$$
\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n + l_n \\
y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + m_n, \quad \forall n \in N
\end{aligned}
$$

where $\{l_n\}_{n=1}^{\infty}, \{m_n\}_{n=1}^{\infty}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$. It is proved that $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed point of S and T. Our result is significant generalization of corresponding result of Ghosh and Debnath [3], Petryshyn and Williamson [7] and Qihou [8].

1. Introduction and preliminaries

Let K be a nonempty subset of a real normed space E. Let T be a self mapping of K. Then T is said to be asymptotically nonexpansive with sequence $\{u_n\} \subset [0,\infty)$ if $\lim_{n\to\infty} u_n = 0$ and

$$\|T^n x - T^n y\| \leq (1+u_n)\|x-y\|$$

for all $x, y \in K$ and $n \geq 1$; and is said to be asymptotically quasi-nonexpansive with sequence $\{u_n\} \subset [0,\infty)$ if $F(T) = \{x \in K : Tx = x\} \neq \phi$, $\lim_{n\to\infty} u_n = 0$ and

$$\|T^n x - x^*\| \leq (1+u_n)\|x - x^*\|$$

for all $x \in K$, $x^* \in F(T)$ and $n \geq 1$.

The mapping T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|$$

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for all $x, y \in K$, and is called quasi-nonexpansive if $F(T) \neq \phi$ and

$$\|Tx - x^*\| \leq \|x - x^*\|$$

for all $x \in K$ and $x^* \in F(T)$. It is therefore clear that a nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive and an asymptotically nonexpansive mapping with a nonempty fixed point set is asymptotically quasi-nonexpansive. The converse do not hold in general.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [1] as an important generalization of the class of nonexpansive maps. They established that if K is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is an asymptotically nonexpansive self mapping of K, then T has a fixed point. In [2], they extended this result to the broader class of uniformly L-Lipschitzian mappings with $L < \lambda$, where λ is sufficiently near 1.

In 1973, Petryshyn and Williamson [7], established a necessary and sufficient condition for a Mann iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. Subsequently, Ghosh and Debnath [3] extended Petryshyn and Williamson's results and established some necessary and sufficient conditions for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mappings. Recently, in [8, 9], Qihou extended the results of Ghosh and Debnath to the more general class of quasi-nonexpansive mapping (i.e. asymptotically quasi-nonexpansive mappings).

The Ishikawa iteration process with errors $\{x_n\}_{n=1}^{\infty}$ (see, e.g., [5]) in K defined as follows

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n + l_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + m_n \end{aligned} \tag{1}$$

where $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$ and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two real sequences in $[0, 1]$ satisfying certain conditions.

It is clear that the Ishikawa iteration process with errors is a generalized case of the Ishikawa iteration process [4], while for all $n \in N$, setting $\beta_n = 0$, it reduces to the Mann iteration process with errors which is a generalized case of the Mann iteration process [6].

Let K be a nonempty subset of a real Banach space X and $S, T : K \to K$ be two asymptotically quasi-nonexpansive mappings. Consider the following modified Ishikawa iteration process with errors $\{x_n\}_{n=1}^{\infty}$ defined as follows:

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n + l_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + m_n \end{aligned} \tag{2}$$

where $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$ and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two real sequences in $[0, 1]$ satisfying certain conditions.

It is clear that if we put $S = T$, then iterative sequences defined by (2) reduces to the sequences defined by (1).

In this paper, we establish a necessary and sufficient conditions for the convergence of the modified Ishikawa iterative sequence with errors involving two asymptotically quasi-nonexpansive mappings defined by (2) to a common fixed point of the mappings defined on a nonempty closed convex subset of a Banach space. Our result is significant generalization of Ghosh and Debnath [3], Petryshyn and Williamson [7] and Qihou [8].

We need the following Lemmas to prove our main result:

Lemma 1.0.1. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be two asymptotically quasi-nonexpansive mappings with sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$ and $F = F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \phi$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S^n y_n + l_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n + m_n, \quad \forall n \in N \end{aligned} \tag{A}$$

where $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$. Then

(a) *$\|x_{n+1} - x^*\| \leq (1 + b_n)\|x_n - x^*\| + t_n$, for all $n \geq 1$, $x^* \in F = F(S) \cap F(T)$, where $b_n = u_n + v_n + u_n v_n$ with $\sum_{n=1}^{\infty} b_n < \infty$ and $t_n = (1 + u_n)\|m_n\| + \|l_n\|$.*

(b) *There exists a constant $M > 0$ such that*

$$\|x_{n+m} - x^*\| \leq M\|x_n - x^*\| + M \sum_{k=n}^{n+m-1} t_k$$

for all $n, m \geq 1$ and $x^* \in F$.

Proof. **(a)** Let $x^* \in F$. Then we have from (A)

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n S^n y_n + l_n - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(S^n y_n - x^*) + l_n\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|S^n y_n - x^*\| + \|l_n\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1+u_n)\|y_n - x^*\| + \|l_n\| \end{aligned} \tag{3}$$

and

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n + m_n - x^*\| \\ &= \|(1-\beta_n)(x_n - x^*) + \beta_n(T^n x_n - x^*) + m_n\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|T^n x_n - x^*\| + \|m_n\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n(1+v_n)\|x_n - x^*\| + \|m_n\| \\ &\leq [1 - \beta_n + \beta_n(1+v_n)]\|x_n - x^*\| + \|m_n\| \\ &\leq (1+v_n)\|x_n - x^*\| + \|m_n\| \end{aligned} \tag{4}$$

from (3) and (4), we have

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n(1+u_n)[(1+v_n)\|x_n-x^*\|+\|m_n\|]+\|l_n\| \\ &\leq [1-\alpha_n+\alpha_n(1+u_n)(1+v_n)]\|x_n-x^*\|+\alpha_n(1+u_n)\|m_n\|+\|l_n\| \\ &\leq [1+\alpha_n(u_n+v_n+u_nv_n)]\|x_n-x^*\|+\alpha_n(1+u_n)\|m_n\|+\|l_n\| \\ &\leq [1+u_n+v_n+u_nv_n]\|x_n-x^*\|+(1+u_n)\|m_n\|+\|l_n\| \\ &\leq (1+b_n)\|x_n-x^*\|+t_n\end{aligned}$$

where $b_n = u_n + v_n + u_nv_n$ with $\sum_{n=1}^{\infty} b_n < \infty$ and $t_n = (1 + u_n)\|m_n\| + \|l_n\|$.
This completes the proof of (a).

(**b**) Since $1 + x \leq e^x$ for all $x > 0$. Then from (a) it can be obtained that

$$\begin{aligned}\|x_{n+m}-x^*\| &\leq (1+b_{n+m-1})\|x_{n+m-1}-x^*\|+t_{n+m-1} \\ &\leq e^{b_{n+m-1}}\|x_{n+m-1}-x^*\|+t_{n+m-1} \\ &\leq e^{(b_{n+m-1}+b_{n+m-2})}\|x_{n+m-2}-x^*\|+e^{b_{n+m-1}}t_{n+m-2}+t_{n+m-1} \\ &\leq e^{(b_{n+m-1}+b_{n+m-2})}\|x_{n+m-2}-x^*\|+e^{b_{n+m-1}}(t_{n+m-1}+t_{n+m-2}) \\ &\leq \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq e^{\sum_{k=n}^{n+m-1} b_k}\|x_n-x^*\|+e^{\sum_{k=n}^{n+m-1} b_k}\sum_{k=n}^{n+m-1} t_k \\ &\leq M\|x_n-x^*\|+M\sum_{k=n}^{n+m-1} t_k, \quad \text{where } M=e^{\sum_{k=n}^{\infty} b_k}\end{aligned}$$

This completes the proof of (b).

Lemma 1.0.2.([9, Lemma 2]) *Let $\{a_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1+r_n)a_n + \beta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. Then
(i) $\lim_{n\to\infty} a_n$ *exists.*
(ii) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

2. Main result

Theorem 2.0.3. *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $S, T : K \to K$ be to asymptotically quasi-nonexpansive mappings (S and T need not be continuous) with sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, and $F = F(S) \cap F(T) = \{x \in K : Sx = Tx = x\} \neq \phi$. Let $\{\alpha_n\}$ and*

$\{\beta_n\}$ be sequences in $[0,1]$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n + l_n$$
$$y_n = (1-\beta_n)x_n + \beta_n T^n x_n + m_n, \quad \forall n \in N$$

where $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are sequences in K satisfying $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$. Then $\{x_n\}$ converges strongly to some common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed point z of S and T. Then

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Conversely, suppose

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Then from Lemma 1.0.1, we have

$$\|x_{n+1} - x^*\| \le (1+b_n)\|x_n - x^*\| + t_n, \quad \forall n \in N, \quad \forall x^* \in F \tag{5}$$

Since $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$, $\sum_{n=1}^{\infty} \|l_n\| < \infty$, $\sum_{n=1}^{\infty} \|m_n\| < \infty$, thus we know $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. So from (5), we obtain

$$d(x_{n+1}, F) \le (1+b_n)d(x_n, F) + t_n,$$

since $\liminf_{n\to\infty} d(x_n, F) = 0$ and from Lemma 1.0.2, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we will show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. For all $\varepsilon > 0$, from Lemma 1.0.1, it can be known that there must exists a constant $M > 0$, such that

$$\|x_{n+m} - x^*\| \le M\|x_n - x^*\| + M \sum_{k=n}^{n+m-1} t_k, \quad \forall x^* \in F, \quad \forall m, n \in N \tag{6}$$

since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{k=n}^{\infty} t_k < \infty$, then there must exists a constant N_1, such that when $n \ge N_1$

$$d(x_n, F) < \frac{\varepsilon_1}{3M}, \quad \text{and} \quad \sum_{k=n}^{\infty} t_k < \frac{\varepsilon_1}{6M}.$$

So there must exists $w^* \in F$, such that

$$d(x_{N_1}, w^*) = \|x_{N_1} - w^*\| < \frac{\varepsilon_1}{3M}.$$

From (6), it can be obtained that when $n \geq N_1$

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - w^*\| + \|x_n - w^*\| \\ &\leq M\|x_{N_1} - w^*\| + M\|x_{N_1} - w^*\| + 2M \sum_{k=N_1}^{\infty} t_k \\ &< M\frac{\varepsilon_1}{3M} + M\frac{\varepsilon_1}{3M} + 2M\frac{\varepsilon_1}{6M} \\ &< \varepsilon_1\end{aligned}$$

that is

$$\|x_{n+m} - x_n\| < \varepsilon_1.$$

This shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and so is convergent, since E is complete. Let $\lim_{n\to\infty} x_n = y^*$. Then $y^* \in K$. It remains to show that $y^* \in F$. Let $\varepsilon_2 > 0$ be given. Then there exists a natural number N_2 such that

$$\|x_n - y^*\| < \frac{\varepsilon_2}{2.max\{2 + u_1, 2 + v_1\}}, \quad \forall n \geq N_2.$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, there must exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$, we have

$$d(x_n, F) < \frac{\varepsilon_2}{3.max\{2 + u_1, 2 + v_1\}},$$

and in particular we have

$$d(x_{N_3}, F) < \frac{\varepsilon_2}{3.max\{2 + u_1, 2 + v_1\}}.$$

Therefore, there exists $z^* \in F$ such that

$$\|x_{N_3} - z^*\| < \frac{\varepsilon_2}{2.max\{2 + u_1, 2 + v_1\}}.$$

Consequently, we have

$$\begin{aligned}\|Sy^* - y^*\| &= \|Sy^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\ &\leq \|Sy^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1 + u_1)\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1 + u_1)\|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1 + u_1)[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (2 + u_1)\|y^* - x_{N_3}\| + (2 + u_1)\|x_{N_3} - z^*\| \\ &< (2 + u_1).\frac{\varepsilon_2}{2.max\{2 + u_1, 2 + v_1\}} + (2 + u_1).\frac{\varepsilon_2}{2.max\{2 + u_1, 2 + v_1\}} \\ &< \varepsilon_2.\end{aligned}$$

This implies that $y^* \in F(S)$.

Similarly, we have

$$\begin{aligned}\|Ty^* - y^*\| &= \|Ty^* - z^* + z^* - x_{N_3} + x_{N_3} - y^*\| \\ &\leq \|Ty^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1+v_1)\|y^* - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1+v_1)\|y^* - x_{N_3} + x_{N_3} - z^*\| + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (1+v_1)[\|y^* - x_{N_3}\| + \|x_{N_3} - z^*\|] + \|z^* - x_{N_3}\| + \|x_{N_3} - y^*\| \\ &\leq (2+v_1)\|y^* - x_{N_3}\| + (2+v_1)\|x_{N_3} - z^*\| \\ &< (2+v_1).\frac{\varepsilon_2}{2.max\{2+u_1, 2+v_1\}} + (2+v_1).\frac{\varepsilon_2}{2.max\{2+u_1, 2+v_1\}} \\ &< \varepsilon_2.\end{aligned}$$

This implies that $y^* \in F(T)$. Hence we conclude that $y^* \in F = F(S) \cap F(T)$, that is, y^* is a common fixed point of S and T. Thus $\{x_n\}_{n=1}^{\infty}$ converges strongly to some common fixed point of S and T. This completes the proof.

Remark 1. If we put $l_n = m_n = 0$, $\forall n \in N$, $S = T$ and $u_n = v_n$, then Theorem 1 of Qihou [8, p.2] is a corollary to the Theorem 2.0.3.

Remark 2. Theorem 2.0.3 contains as special cases of the main result of Qihou [8, Theorem 1, p.2] together with [8, Corollaries 1 and 2], which are themselves extensions of the results of Ghosh and Debnath [3] Petryshyn and Williamson [7].

Remark 3. Theorem 2.0.3 remains true for the subclass of asymptotically nonexpansive mappings.

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