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THE SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE CARLSON-SHAFFER OPERATOR

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Abstract. In this paper a new class of analytic functions, associated with the Carlson-Shaffer operator, is investigated. The sharp estimate for the Second Hankel determinant and class preserving transforms are studied.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions in the *open* unit disc

$$\mathbb{U} := \{z : z \in \mathbb{C}, \quad |z| < 1\}.$$

We denote by $\mathscr{A}_0$, the subclass of $\mathscr{A}$ consisting of *normalized* functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

For the functions f and g in $\mathscr{A}$ given by the series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \qquad (z \in \mathbb{U}),$$

the *Hadamard product* (or *Convolution*) $f * g$, is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \qquad (z \in \mathbb{U}).$$

The function $f * g \in \mathscr{A}$. We recall that the Carlson- Shaffer operator [3]

$$\mathscr{L}(\alpha, \beta) : \mathscr{A}_0 \to \mathscr{A}_0 \qquad (\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- : \{0, -1, -2, \dots\})$$

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is defined by:

$$\mathscr{L}(\alpha,\beta)f(z) = \phi(\alpha,\beta;z) * f(z) \quad (z \in \mathbb{U},\ f \in \mathscr{A}), \tag{1.2}$$

where

$$\phi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} z^{k+1} \quad (z \in \mathbb{U}) \tag{1.3}$$

and $(\lambda)_k$ is the *Pochhammer symbol* (or *shifted factorial*) defined in terms of the *Gamma function* by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} := \begin{cases} 1 & (k=0) \\ \lambda(\lambda+1)\dots(\lambda+k-1) & (k \in \mathbb{N} = \{1,2,\dots\}). \end{cases}$$

It can be readily verified that $\mathscr{L}(\alpha,\alpha)$ is the *identity* operator; the operators $\mathscr{L}(\alpha,\beta)$ and $\mathscr{L}(\gamma,\delta)$ *commute*, that is

$$\mathscr{L}(\alpha,\beta)\mathscr{L}(\gamma,\delta)f(z) = \mathscr{L}(\gamma,\delta)\mathscr{L}(\alpha,\beta)f(z) \quad (f \in \mathscr{A}_0)$$

and the following *transitive* property holds true:

$$\mathscr{L}(\alpha,\beta)\mathscr{L}(\beta,\gamma)f = \mathscr{L}(\alpha,\gamma)f \quad (\beta,\gamma \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ f \in \mathscr{A}_0).$$

In the particular case $\alpha = 2,\ \beta = 1$, the operator $\mathscr{L}(\alpha,\beta)$ reduces to the *Alexander's transform*:

$$\mathscr{L}(2,1)f(z) = zf'(z) \quad (f \in \mathscr{A}_0).$$

Moreover, the popular *Owa-Srivastava fractional differential operator*

$$\Omega_z^\lambda : \mathscr{A}_0 \to \mathscr{A}_0 \quad (0 \le \lambda < 1,\ z \in \mathbb{U})$$

is related to the Carlson-Shaffer operator by the formula:

$$\Omega_z^\lambda f(z) = \mathscr{L}(2,2-\lambda)f(z)$$

(see [23, 24, 25], also see [18, 19]). By using the Carlson-Shaffer operator we introduce the following class of functions:

Definition 1. The function $f \in \mathscr{A}_0$ is said to be in the class $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$ $\quad (-\frac{\pi}{2} < \theta < \frac{\pi}{2},\ 0 \le \rho < 1,\ \alpha \in \mathbb{C},\ \beta \in \mathbb{C}\setminus\mathbb{Z}_0^-)$ if

$$\Re\left\{e^{i\theta}\frac{\mathscr{L}(\alpha,\beta)f(z)}{z}\right\} > \rho\cos\theta \quad (z \in \mathbb{U}). \tag{1.4}$$

The class $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$ generalizes several well known subclasses of $\mathscr{A}_0$. For example, taking $\alpha=\beta$; $\alpha=2,\beta=1$ and $\alpha=2,\ \beta=2-\lambda\quad(0\le\lambda<1)$ respectively, we get the following interesting classes:

$$\mathscr{R}_{\alpha,\alpha}(\theta,\rho)=\left\{f\in\mathscr{A}_0:\Re\left(e^{i\theta}\frac{f(z)}{z}\right)>\rho\ \cos\theta\right\}:=\mathscr{R}_0(\theta,\rho) \tag{1.5}$$

$$\bigcup_\theta \mathscr{R}_0(\theta,\rho)=\mathscr{R}_0(\rho),$$

$$\mathscr{R}_{2,1}(\theta,\rho)=\left\{f\in\mathscr{A}_0:\Re\left(e^{i\theta}f'(z)\right)>\rho\ cos\theta\right\}:=\mathscr{R}_1(\theta,\rho)$$

$$\bigcup_\theta \mathscr{R}_1(\theta,\rho)=\mathscr{R}_1(\rho)$$

and

$$\mathscr{R}_{2,2-\lambda}(\theta,\rho)=\left\{f\in\mathscr{A}_0:\Re\left(e^{i\theta}\frac{\Omega_z^\lambda f(z)}{z}\right)>\rho\ cos\theta\right\}:=\mathscr{R}_\lambda(\theta,\rho)$$

$$\bigcup_\theta \mathscr{R}_\lambda(\theta,\rho)=\mathscr{R}_\lambda(\rho).$$

It is well known that the functions in the class $\mathscr{R}_1(\rho)$ are univalent close-to-convex [4]. Moreover, if $0\le\mu<\lambda<1$ then

$$\mathscr{R}_1(\rho)\subset\mathscr{R}_\lambda(\rho)\subset\mathscr{R}_\mu(\rho)\subset\mathscr{R}_0(\rho)$$

(cf. [16, 20]). For initial seminal work on the class $\mathscr{R}_1(0):=\mathscr{R}_1$ one may see the classical paper of Macgregor [17]. The family of functions $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$ is characterized by the following function class:

$$\mathscr{P}:=\{p\in\mathscr{A}:\ p(0)=1,\ \Re(p(z))>0,\ z\in\mathbb{U}\}.$$

Infact, it follows from (1.4) that the function $f\in\mathscr{A}_0$ is in the class $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$ if and only if

$$e^{i\theta}\frac{\mathscr{L}(\alpha,\beta)f(z)}{z}=[(1-\rho)p(z)+\rho]\cos\theta+i\ \sin\theta \tag{1.6}$$

for some function $p\in\mathscr{P}$.

For the complex sequence $a_n,a_{n+1},a_{n+2},\ldots$, the *Hankel matrix*, named after Herman Hankel (1839-1873), is the infinite matrix whose $(i,j)^{th}$ entry a_{ij} is defined by

$$a_{ij}=a_{n+i+j-2}\quad(i,j,n\in\mathbb{N}).$$

The q^{th} Hankel matrix $(q\in\mathbb{N}\setminus\{1\})$, is by definition, the following $q\times q$ square sub matrix:

$$\begin{pmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q}\\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots & a_{n+q+1}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n+q-1} & a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-1}\end{pmatrix}$$

We observe that the Hankel matrix has constant positive slopping diagonals whose entries also satisfy:

$$a_{ij} = a_{i-1,j+1} \quad (i \in \mathbb{N} \setminus \{1\};\ j \in \mathbb{N}).$$

This also describes the Hankel matrix without reference to a particular sequence. The determinant of the q^{th} Hankel matrix, usually denoted by $H_q(n)$, is called the q^{th} Hankel determinant (cf. [22]). In the particular cases $q = 2,\ n = 1,\ a_1 = 1$ and $q = 2,\ n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = a_2 a_4 - a_3^2.$$

We refer to $H_2(2)$ as the *Second Hankel determinant.*

It is fairly well known that for the univalent function of the form (1.1) the sharp inequality $|H_2(1)| = |a_3 - a_2^2| \leq 1$ holds true [4]. For a family $\Im$ of functions in $\mathscr{A}_0$, the more general problem of finding sharp estimates for the functional $|\mu a_2^2 - a_3|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the *Fekete-Szegö problem* for $\Im$. The Fekete-Szegö problem for the families of univalent functions, starlike functions, convex functions, close-to-convex functions has been completely settled in [5, 11, 12, 13]. For related results also see [19].

Recently Janteng et.al.[8] and the first author and Gochhayat [20] obtained sharp estimates on the Second Hankel determinant for the families $\mathscr{R}_1(\rho)$ and $\mathscr{R}_\lambda(\theta,\rho)$ respectively. For some more recent work see [1, 6, 7, 9, 10, 21]. In this paper we generalize the results of [8] and [20] by finding sharp bounds for $|H_2(2)| = |a_2 a_4 - a_3^2|$ for f in $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$. We also obtain here some basic properties such as class preserving transforms for the class $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$.

2. Preliminaries

Each of the following results will be required in our present investigation:

Lemma 2.1. *(cf. [4]) Let the function $p \in \mathscr{P}$ be given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad (z \in \mathbb{U}). \tag{2.1}$$

Then,

$$|c_k| \leq 2 \qquad (k \in \mathbb{N}). \tag{2.2}$$

The estimate (2.2) is sharp.

Lemma 2.2. *(cf.[15], p.254, also see [14]) Let the function $p \in \mathscr{P}$ be given by the power series (2.1). Then,*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.3}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{2.4}$$

for some complex numbers $x,\ z$ satisfying $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.3. *(cf. [26]) Let f and g be univalent convex functions in $\mathbb{U}$. Then, $f * g$ is also a univalent convex function in $\mathbb{U}$.*

Lemma 2.4. *(cf. [26], also see [16]) Let f and g be starlike of order 1/2. Then, for each function $\mathscr{F}(z)$ satisfying $\Re(\mathscr{F}(z)) > \lambda \quad (0 \leq \lambda < 1,\ z \in \mathbb{U})$,*

$$\Re\left(\frac{f(z) * \mathscr{F}(z)g(z)}{f(z) * g(z)}\right) > \lambda \quad (z \in \mathbb{U}). \tag{2.5}$$

3. Main results

We state and prove the following:

Theorem 3.1. *Let the function f, given by (1.1), be in the class $\mathscr{R}_{\alpha,\beta}(\theta,\rho) \quad (0 \leq \rho < 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2})$. If $0 < \beta < 2,\ 0 < \beta \leq \alpha < \frac{2+5\beta}{2-\beta}$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{4\beta^2(\beta+1)^2(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)^2}. \tag{3.1}$$

The estimate (3.1) is sharp.

Proof. Let $f \in \mathscr{R}_{\alpha,\beta}(\theta,\rho) \quad (0 \leq \rho < 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Then using (1.2), (1.3), and (1.6) we write

$$\begin{aligned} e^{i\theta}\frac{\mathscr{L}(\alpha,\beta)f(z)}{z} &= e^{i\theta}\left[1 + \sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_n z^{n-1}\right] \\ &= \left[(1-\rho)p(z)+\rho\right]\cos\theta + i\sin\theta \end{aligned} \tag{3.2}$$

where $p \in \mathscr{P}$ and is given by (2.1).

A comparison of the coefficients, in (3.2) gives

$$\begin{aligned} \frac{\alpha\ e^{i\theta}}{\beta}a_2 &= c_1(1-\rho)\cos\theta, \\ \frac{(\alpha)_2\ e^{i\theta}}{(\beta)_2}a_3 &= c_2(1-\rho)\cos\theta, \\ \frac{(\alpha)_3\ e^{i\theta}}{(\beta)_3}a_4 &= c_3(1-\rho)\cos\theta. \end{aligned} \tag{3.3}$$

Therefore, (3.3) yields the following:

$$|a_2 a_4 - a_3^2| = \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)}\left|\frac{\beta+2}{\alpha+2}c_1c_3 - \frac{\beta+1}{\alpha+1}c_2^2\right|.$$

Since the functions $p(z)$ and $p(e^{i\theta}z) \quad (\theta \in \mathbb{R})$ are members of the class $\mathscr{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c \ \ (c \in [0,2])$.

Using Lemma 2.2, we get

$$
\begin{aligned}
|a_2a_4-a_3^2| &= \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)} \\
&\quad\times\Big|\frac{\beta+2}{4(\alpha+2)}c\{c^3+2(4-c^2)cx-c(4-c^2)x^2+2(4-c^2)(1-|x|^2)z\} \\
&\quad-\frac{\beta+1}{4(\alpha+1)}\{c^2+x(4-c^2)\}^2\Big| \\
&= \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)}\Big|\frac{\beta+2}{4(\alpha+2)}c^4+\frac{(\beta+2)(4-c^2)c^2}{2(\alpha+2)}x-\frac{(\beta+2)(4-c^2)c^2}{4(\alpha+2)}x^2 \\
&\quad+\frac{(\beta+2)(4-c^2)c(1-|x|^2)z}{2(\alpha+2)}-\frac{\beta+1}{4(\alpha+1)}c^4-\frac{(\beta+1)(4-c^2)c^2}{2(\alpha+1)}x-\frac{(\beta+1)(4-c^2)^2}{4(\alpha+1)}x^2\Big|. \\
&= \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)}\Big|\frac{(\alpha-\beta)c^4}{4(\alpha+1)_2}+\frac{(4-c^2)c^2(\alpha-\beta)}{2(\alpha+1)_2}x \\
&\quad-\frac{4-c^2}{4}\left(\frac{c^2(\beta+2)}{(\alpha+2)}+\frac{(\beta+1)(4-c^2)}{(\alpha+1)}\right)x^2+\frac{c(4-c^2)(\beta+2)(1-|x|^2)}{2(\alpha+2)}z\Big|.
\end{aligned}
$$

An application of triangle inequality and replacement of $|x|$ by μ, give

$$
\begin{aligned}
|a_2a_4-a_3^2| &\le \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)}\Big[\frac{(\alpha-\beta)c^4}{4(\alpha+1)_2}+\frac{(4-c^2)c^2(\alpha-\beta)}{2(\alpha+1)_2}\mu \\
&\quad+\frac{4-c^2}{4}\left(\frac{c^2(\beta+2)}{(\alpha+2)}+\frac{(\beta+1)(4-c^2)}{(\alpha+1)}\right)\mu^2+\frac{c(4-c^2)(\beta+2)}{2(\alpha+2)}-\frac{c(4-c^2)(\beta+2)}{2(\alpha+2)}\mu^2\Big]. \\
&= \frac{\beta^2(\beta+1)(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)}\Big[\frac{(\alpha-\beta)c^4}{4(\alpha+1)_2}+\frac{c(4-c^2)(\beta+2)}{2(\alpha+2)} \\
&\quad+\frac{(4-c^2)c^2(\alpha-\beta)}{2(\alpha+1)_2}\mu+\frac{(4-c^2)(2-c)\{2(\alpha+2)(\beta+1)-c(\alpha-\beta)\}}{4(\alpha+1)_2}\mu^2\Big] \\
&= \mathscr{F}(c,\mu) \quad \text{(say)} \qquad (3.4)
\end{aligned}
$$

where $0\le c\le 2$ and $0\le\mu\le 1$.

We next maximize the function $\mathscr{F}(c,\mu)$ on the closed rectangle $[0,2]\times[0,1]$. We first assume that $0<\beta<\alpha$. A routine calculation gives

$$
\begin{aligned}
\frac{\partial\mathscr{F}}{\partial\mu} &= \frac{\beta^2(\beta+1)(1-\rho)^2(4-c^2)(\alpha-\beta)\cos^2\theta}{2\alpha^2(\alpha+1)^2(\alpha+2)}\left[c^2+(c-2)\left\{c-\frac{2(\alpha+2)(\beta+1)}{(\alpha-\beta)}\right\}\mu\right] \\
&= \frac{\beta^2(\beta+1)(1-\rho)^2(4-c^2)(\alpha-\beta)\cos^2\theta}{2\alpha^2(\alpha+1)^2(\alpha+2)}\left[c^2+(c-2)\left\{\frac{\alpha(c-2)-\beta(c+4)-2\alpha\beta-4}{(\alpha-\beta)}\right\}\mu\right].
\end{aligned}
$$

Therefore, for $0<c<2$ and $0<\mu<1$, we have $\frac{\partial\mathscr{F}}{\partial\mu}>0$. Thus $\mathscr{F}(c,\mu)$ cannot have a maximum in the interior of the closed rectangle $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$,

$$\max_{0\le\mu\le1}\mathscr{F}(c,\mu)=\mathscr{F}(c,1).$$

Set

$$\mathscr{F}(c,1)=\mathscr{H}(c) \quad \text{(say)}.$$

Again a routine calculation gives

$$\mathscr{H}'(c)=\frac{-8c(\alpha-\beta)\beta^2(\beta+1)(1-\rho)^2\cos^2\theta\left[c^2+\frac{\alpha(\beta-2)+5\beta+2}{\alpha-\beta}\right]}{4\alpha^2(\alpha+1)^2(\alpha+2)}.$$

Since $0<\beta<\alpha<\frac{2+5\beta}{2-\beta}$, we get $\left[c^2+\frac{\alpha(\beta-2)+5\beta+2}{\alpha-\beta}\right]>0$, so that $\mathscr{H}'(c)<0$ for $0<c<2$. Also $\mathscr{H}(c)>\mathscr{H}(2)$. Therefore $\max_{0\le c\le 2}\mathscr{H}(c)$ occurs at $c=0$ and the upper bound of (3.4) corresponds to $\mu=1$ and $c=0$. Next, taking $0<\beta=\alpha$ in (3.4) we have

$$\mathscr{F}(c,\mu)=\frac{(1-\rho)^2(4-c^2)}{2}\left\{c+(2-c)\mu^2\right\}\cos^2\theta,$$

$$\frac{\partial\mathscr{F}}{\partial\mu}=(1-\rho)^2(4-c^2)(2-c)\mu\cos^2\theta>0$$

and

$$\mathscr{H}(c)=(1-\rho)^2(4-c^2)\cos^2\theta.$$

Hence, the maximum of $\mathscr{F}(c,\mu)$ occurs at $c=0$ and $\mu=1$.
Therefore,

$$|a_2a_4-a_3^2|\le\frac{4\beta^2(\beta+1)^2(1-\rho)^2\cos^2\theta}{\alpha^2(\alpha+1)^2}.$$

This is the assertion of (3.1). Equality holds for the function

$$f(z)=\phi(\beta,\alpha;z)*e^{-i\theta}\left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\cos\theta+i\sin\theta\right)\right].$$

The proof of the theorem is completed. □

Taking $0<\beta=\alpha$ in Theorem 3.1 we have the following:

Corollary 3.2. *Let the function f given by (1.1), be a member of the class $\mathscr{R}_0(\rho)$ $(0\le\rho<1)$. Then*

$$|a_2a_4-a_3^2|\le 4(1-\rho)^2.$$

Equality holds for the function

$$z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\right).$$

Taking $\alpha=2$ and $\beta=1$ Theorem 3.1 we get the following:

Corollary 3.3. *Let the function f, given by (1.1) be in the class $\mathscr{R}_1(\rho)$ $(0 \le \rho < 1)$. Then*

$$|a_2 a_4 - a_3^2| \le \frac{4(1-\rho)^2}{9}.$$

Equality holds for the function

$$f(z) = \phi(1,2;z) * \left\{ z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\right)\right\}.$$

The choice ρ = 0 in Corollary 3.3 gives a result of Janteng et. al. [8] for the class $\mathscr{R}_1$.
Similarly the choice $\alpha = 2,\ \beta = 2-\lambda, (0 \le \lambda \le 1)$, in Theorem 3.1 gives the following recent result of Mishra and Gochhayat [20] for the class $\mathscr{R}_\lambda(\theta,\rho)$.

Corollary 3.4. *Let the function f, given by (1.1), be in the class $\mathscr{R}_\lambda(\theta,\rho)$, $(0 \le \lambda \le 1,\ 0 \le \rho < 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Then*

$$|a_2 a_4 - a_3^2| \le \frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2\cos^2\theta}{9}.$$

The estimate is sharp, for the function

$$f(z) = \phi(2-\lambda,2;z) * e^{-i\theta}\left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\right)\cos\theta + i\sin\theta\right].$$

Corollary 3.5. *Let the function f, given by (1.1), be a member of the class $\mathscr{R}_{\alpha,\beta}(\rho)$ $(0 < \beta < 2$, $0 < \beta < \alpha < \dfrac{2+5\beta}{2-\beta})$. Then*

$$|a_2 a_4 - a_3^2| \le \frac{4\beta^2(\beta+1)^2(1-\rho)^2}{\alpha^2(\alpha+1)^2}.$$

Equality holds for the function

$$f(z) = \phi(\beta,\alpha;z) * \left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\right)\right].$$

Theorem 3.6. *Let $f \in \mathscr{S}^*(1/2)$ and $g \in \mathscr{R}_{\alpha,\beta}(\theta,\rho)$ $\left(0 \le \rho < 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$. Then*

$$f * g \in \mathscr{R}_{\alpha,\beta}(\theta,\rho).$$

Proof. Since Hadamard product is associative and commutative, we have

$$\mathscr{L}(\alpha,\beta)(f*g)(z) = f(z) * \mathscr{L}(\alpha,\beta)g(z).$$

Therefore,

$$\frac{e^{i\theta}\mathscr{L}(\alpha,\beta)(f*g)(z)}{z} = \frac{f(z) * \frac{e^{i\theta}\mathscr{L}(\alpha,\beta)g(z)}{z} z}{f(z)*z}.$$

Now applying Lemma 2.4, we get

$$\Re\left(\frac{e^{i\theta}\mathscr{L}(\alpha,\beta)(f*g)(z)}{z}\right) > \rho\cos\theta.$$

Hence $f * g \in \mathscr{R}_{\alpha,\beta}(\theta,\rho)$ and the proof of Theorem 3.6 is completed. ☐

Theorem 3.7. *Let* $f \in \mathscr{R}_{\alpha,\beta}(\theta,\rho)$ $\left(0 \le \rho < 1,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$. *Then, the function* $\mathscr{J}(f)$ *defined by the integral transform.*

$$\mathscr{J}(f)(z) = \frac{\gamma+1}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)\,dt \quad (z \in \mathbb{U},\ \gamma > -1)$$

is also in $\mathscr{R}_{\alpha,\beta}(\theta,\rho)$.

Proof. The integral transform $\mathscr{J}(f)$ can be written in terms of the Carlson-Shaffer operator as

$$\mathscr{J}(f)(z) = (\mathscr{L}(\gamma+1,\gamma+2)f)(z).$$

Hence

$$(\mathscr{L}(\alpha,\beta)\mathscr{J}(f))(z) = \mathscr{L}(\gamma+1,\gamma+2)\mathscr{L}(\alpha,\beta)f(z) = \phi(\gamma+1,\gamma+2;z) * \mathscr{L}(\alpha,\beta)f(z).$$

Therefore,

$$\frac{e^{i\theta}(\mathscr{L}(\alpha,\beta)\mathscr{J}(f))(z)}{z} = \frac{\phi(\gamma+1,\gamma+2;z) * \left(e^{i\theta}\mathscr{L}(\alpha,\beta)f(z)/z\right)z}{\phi(\gamma+1,\gamma+2;z) * z}.$$

Using a result of Bernardi [2], it can be verified that $\phi(\gamma+1,\gamma+2;z) \in \mathscr{S}^*(1/2)$. Thus by applying Lemma 2.4, the proof of Theorem 3.7 is completed. □

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