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ON SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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Abstract. Some inequalities for twice differentiable mappings are presented. Some applications to special means of real numbers are also given.

1. Introduction

A function $f : I \subset R \to R$ is called convex on the interval I of real numbers if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all points x and y in I and all $\lambda \in [0,1]$.

Let $f : I \subset R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

is known as Hermite-Hadamard's inequality for convex functions [7].

In 1935, G. Grüss (see [7, p.70]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

Theorem A. *Let $h, g : [a,b] \to R$ be two integrable functions such that $\phi_1 \le h(x) \le \Phi_1$ and $\gamma_1 \le g(x) \le \Gamma_1$ for all $x \in [a,b]$, where $\phi_1, \Phi_1, \gamma_1, \Gamma_1$, are real numbers. Then we have*

$$|T(h,g)| := \left|\frac{1}{b-a}\int_a^b h(x)g(x)dx - \frac{1}{b-a}\int_a^b h(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi_1-\phi_1)(\Gamma_1-\gamma_1)$$

and the inequality is sharp, in the sense that the constant $1/4$ cannot be replaced by a smaller one.

In [7, p.40], Chebychev's inequlity is given by the following theorem:

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Theorem B. *Let f and g be two functions which are integrable and monotone in the same sense on (a,b) and let p be a positive and integrable function on the same interval. Then,*

$$\int_a^b p(x)f(x)g(x)dx\int_a^b p(x)dx \geq \int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx$$

with equality if and only if one of the functions f,g reduces to a constant.

If f and g are monotone in the opposite sense, the reverse inequality holds.

Let $f: I\subset R\to R$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. The inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

is known as Bullen's inequality for convex functions [1, p.39].

In the literature, the following definition is well known:

Let $f:[a,b]\to R$ and $p\in R^+$. The p-norm of the function f on $[a,b]$ is defined by

$$\|f\|_p \begin{cases} \left(\int_a^b |f(x)|^p dx\right)^{1/p}, & 0<p<\infty, \\ \sup|f(x)|, & p=\infty, \end{cases}$$

and $L^p([a,b])$ is the set of all functions $f:[a,b]\to R$ such that $\|f\|_p<\infty$.

For several recent results concerning Hermite-Hadamard's inequality and twice differentiable mappings, we refer the reader to [1]-[6].

In this paper, we give new inequalities for twice differentiable mappings and some applications to special means of real numbers.

2. Main results

We first prove the following lemma:

Lemma 1. *Let $f: I\subset R\to R$ be twice differentiable mapping on I^0 such that $f"\in L'[a,b]$, where $a,b\in I^0$ with $a<b$. Then we have the equality*

$$\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)f"(x)dx+\int_{\frac{a+b}{2}}^b(b-x)\left(x-\frac{a+b}{2}\right)f"(x)dx\right]$$
$$=\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx \qquad (1)$$

where, I^0 denotes the interior of I.

Proof. By integration by parts twice, we have

$$\int_a^{\frac{a+b}{2}}(x-a)\Big(\frac{a+b}{2}-x\Big)f''(x)dx+\int_{\frac{a+b}{2}}^b(b-x)\Big(x-\frac{a+b}{2}\Big)f''(x)dx$$
$$=\int_a^{\frac{a+b}{2}}\Big[2x-\Big(\frac{a+b}{2}+a\Big)\Big]f'(x)dx+\int_{\frac{a+b}{2}}^b\Big[2x-\Big(\frac{a+b}{2}+b\Big)\Big]f'(x)dx$$
$$=(b-a)\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big]-2\int_a^b f(x)dx.$$

Hence, we obtain desired equality (1). □

Theorem 1. *Let $f: I\subset R\to R$ be twice differentiable mapping on I^0 such that $f''\in L'[a,b]$, where $a,b\in I^0$ with $a<b$. If the mapping*

$$\varphi(x)=\begin{cases}(x-a)\Big(\frac{a+b}{2}-x\Big)f''(x), & x\in\Big[a,\frac{a+b}{2}\Big),\\ (b-x)\Big(x-\frac{a+b}{2}\Big)f''(x), & x\in\Big[\frac{a+b}{2},b\Big],\end{cases}$$

is convex on $[a,b]$, then we have the inequality

$$\frac{(b-a)^2}{64}\Big[f''\Big(\frac{3a+b}{4}\Big)+f''\Big(\frac{a+3b}{4}\Big)\Big]\le\frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big]-\frac{1}{b-a}\int_a^b f(x)dx$$
$$\le\frac{(b-a)^2}{128}\Big[f''\Big(\frac{3a+b}{4}\Big)+f''\Big(\frac{a+3b}{4}\Big)\Big].\tag{2}$$

Proof. Applying the first inequality of Hermite-Hadamard for the mapping φ, we write

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx\ge\varphi\Big(\frac{3a+b}{4}\Big)=\frac{(b-a)^2}{16}f''\Big(\frac{3a+b}{4}\Big),$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\varphi(x)dx\ge\varphi\Big(\frac{a+3b}{4}\Big)=\frac{(b-a)^2}{16}f''\Big(\frac{a+3b}{4}\Big).$$

Applying the Bullen's inequality for the mapping φ, we have

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\varphi(x)dx\le\frac{1}{2}\Big[\varphi\Big(\frac{3a+b}{4}\Big)+\frac{\varphi(a)+\varphi\Big(\frac{a+b}{2}\Big)}{2}\Big]=\frac{(b-a)^2}{32}f''\Big(\frac{3a+b}{4}\Big),$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\varphi(x)dx\le\frac{1}{2}\Big[\varphi\Big(\frac{a+3b}{4}\Big)+\frac{\varphi\Big(\frac{a+b}{2}\Big)+\varphi(b)}{2}\Big]=\frac{(b-a)^2}{32}f''\Big(\frac{a+3b}{4}\Big).$$

Adding all these inequalities and from Lemma 1, we have

$$\frac{(b-a)^2}{64}\Big[f''\Big(\frac{3a+b}{4}\Big)+f''\Big(\frac{a+3b}{4}\Big)\Big]\le\frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big]-\frac{1}{b-a}\int_a^b f(x)dx$$

$$\le \frac{(b-a)^2}{128}\left[f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]$$

which is required inequality (2). □

Example. Let us consider function f defined as $f(x)=x^3$. Then we have $f''(x)=6x$. If we apply Theorem 1, we obtain

$$\frac{3(b-a)^2(a+b)}{32}\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\le\frac{3(b-a)^2(a+b)}{64}.$$

If we choose $a=-2$, $b=1$, we deduce

$$-\frac{27}{32}\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\le-\frac{27}{64}$$

where

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx=-\frac{9}{16}.$$

Theorem 2. *Let $f: I\subset R\to R$ be twice differentiable mapping on I^0, $p>1$, and $q=\frac{p}{p-1}$. If $|f''|\in L'[a,b]$, where $a,b\in I^0$ with $a<b$, then we have the inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^{(p+1)/p}}{2^{(2p+1)/p}}[B(p+1,p+1)]^{1/p}\|f''\|_q$$

where, $B(p,q)$ is Euler's Beta function.

Proof. By (1) and Hölder's inequality, we have that

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\left|\frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)f''(x)dx+\int_{\frac{a+b}{2}}^b(b-x)\left(x-\frac{a+b}{2}\right)f''(x)dx\right]\right|\\
&\le\frac{1}{2(b-a)}\left[\left(\int_a^{\frac{a+b}{2}}(x-a)^p\left(\frac{a+b}{2}-x\right)^p\right)^{1/p}\left(\int_a^{\frac{a+b}{2}}|f''(x)|^q dx\right)^{1/q}\right.\\
&\quad\left.+\left(\int_{\frac{a+b}{2}}^b(b-x)^p\left(x-\frac{a+b}{2}\right)^p\right)^{1/p}\left(\int_{\frac{a+b}{2}}^b|f''(x)|^q dx\right)^{1/q}\right].
\end{aligned}$$

Using the change of the variable $x=(1-t)a+t\left(\frac{a+b}{2}\right)$ and from $dx=\left(\frac{b-a}{2}\right)dt$, we write

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}(x-a)^p\left(\frac{a+b}{2}-x\right)^p dx\\
&=\left(\frac{b-a}{2}\right)\int_0^1\left((1-t)a+t\left(\frac{a+b}{2}\right)-a\right)^p\left(\frac{a+b}{2}-(1-t)a-t\left(\frac{a+b}{2}\right)\right)^p dt\\
&=\left(\frac{b-a}{2}\right)^{2p+1}\int_0^1 t^p(1-t)^p dt=\left(\frac{b-a}{2}\right)^{2p+1}B(p+1,p+1)
\end{aligned}$$

where, $B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt,\ (p,q>0)$.

Using the change of the variable $x=(1-t)\frac{a+b}{2}+tb$ and from $dx=\left(\frac{b-a}{2}\right)dt$, we write

$$\int_{\frac{a+b}{2}}^{b}(b-x)^p\left(x-\frac{a+b}{2}\right)^p dx=\left(\frac{b-a}{2}\right)\int_0^1\left(b-(1-t)\frac{a+b}{2}-tb\right)^p\left((1-t)\frac{a+b}{2}+tb-\frac{a+b}{2}\right)^p dt$$
$$=\left(\frac{b-a}{2}\right)^{2p+1}\int_0^1 t^p(1-t)^p dt=\left(\frac{b-a}{2}\right)^{2p+1}B(p+1,p+1).$$

Combining all this inequalities, we have

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^{(p+1)/p}}{2^{(2p+1)/p}}[B(p+1,p+1]^{1/p}\|f''\|_q.$$

Hence, the theorem is proved. □

Remark. Let f be as in Theorem 2. For $p>1$, $p\in N$ and using the equality

$$B(p+1,p+1)=\frac{p!}{(p+1)\cdots(2p+1)}=\frac{[p!]^2}{(2p+1)!}$$

we deduce

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^{(p+1)/p}}{2^{(2p+1)/p}}\left[\frac{[p!]^2}{(2p+1)!}\right]^{1/p}\|f''\|_q$$

which gives for $p=2$ that

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{\sqrt{960}}{960}(b-a)^{3/2}\|f''\|_2.$$

Theorem 3. *Let $f:I\subset R\to R$ be twice differentiable mapping on I^0 such that $\gamma\le f''(x)\le\mu$ on $[a,\frac{a+b}{2})\subset I^0$ and $\gamma'\le f''(x)\le\mu'$ on $[\frac{a+b}{2},b]\subset I^0$. If $f''\in L'[a,b]$, then we have the inequality*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{96}[f'(b)-f'(a)]\right|\le\frac{(b-a)^2}{128}[(\mu-\gamma)+(\mu'-\gamma')] \tag{3}$$

Proof. By Grüss inequality, we have that

$$\left|\Delta_1-\left(\frac{1}{b-a}\right)^2\left[\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)dx\int_a^{\frac{a+b}{2}}f''(x)dx+\int_{\frac{a+b}{2}}^b(b-x)\left(x-\frac{a+b}{2}\right)dx\int_{\frac{a+b}{2}}^b f''(x)dx\right]\right|$$
$$\le\frac{1}{4}[(\ell-l)(\mu-\gamma)+(\ell'-l')(\mu'-\gamma')]$$

where

$$\ell=\sup_{x\in[a,\frac{a+b}{2}]}\left\{(x-a)\left(\frac{a+b}{2}-x\right)\right\}=\frac{(b-a)^2}{16},\qquad l=\inf_{x\in[a,\frac{a+b}{2}]}\left\{(x-a)\left(\frac{a+b}{2}-x\right)\right\}=0,$$

$$\ell' = \sup_{x\in[\frac{a+b}{2},b]}\left\{(b-x)\left(x-\frac{a+b}{2}\right)\right\} = \frac{(b-a)^2}{16}, \qquad l' = \inf_{x\in[\frac{a+b}{2},b]}\left\{(b-x)\left(x-\frac{a+b}{2}\right)\right\} = 0.$$

By integration by parts, we get

$$\int_a^{\frac{a+b}{2}} (x-a)\left(\frac{a+b}{2}-x\right)dx = \int_{\frac{a+b}{2}}^b (b-x)\left(x-\frac{a+b}{2}\right)dx = \frac{(b-a)^3}{48}.$$

By Lemma 1, we have that

$$\Delta = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

where $\Delta = \frac{1}{2}\Delta_1$. Hence, we obtain,

$$\begin{aligned}&\left|\Delta-\frac{1}{2}\left(\frac{1}{(b-a)^2}\right)\frac{(b-a)^3}{48}\left\{\left[f'\left(\frac{a+b}{2}\right)-f'(a)\right]+\left[f'(b)-f'\left(\frac{a+b}{2}\right)\right]\right\}\right|\\ &\le \frac{(b-a)^2}{128}[(\mu-\gamma)+(\mu'-\gamma')].\end{aligned}$$

That is,

$$\left|\Delta-\frac{b-a}{96}[f'(b)-f'(a)]\right| \le \frac{(b-a)^2}{128}[(\mu-\gamma)+(\mu'-\gamma')]$$

and the theorem is proved. □

Theorem 4. *Let $f: I\subset R\to R$ be twice differentiable mapping on I^0. If $f'' \in L'[a,b]$ and $k \le f''(x) \le K$ for all $x\in[a,b]\subset I^0$, then we have the inequality*

$$K\frac{(b-a)^2}{48}-\frac{1}{2(b-a)}\left(\frac{\sigma_1\sigma_2}{\sigma_3}+\frac{\sigma_4\sigma_5}{\sigma_6}\right) \le I_1+I_2 \le k\frac{(b-a)^2}{48}-\frac{1}{2(b-a)}\left(\frac{\alpha_1\alpha_2}{\alpha_3}+\frac{\alpha_4\alpha_5}{\alpha_6}\right)$$

where,

$$I_1+I_2 = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

and $\alpha_i, \sigma_i, (i=1,\dots,6)$ are given by (12)–(17) and (18)–(23) respectively.

Proof. By Lemma 1, we have that

$$\Delta = I_1+I_2 = \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx$$

where

$$\begin{aligned}I_1 &= \frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}} (x-a)\left(\frac{a+b}{2}-x\right)f''(x)dx,\\ I_2 &= \frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^b (b-x)\left(x-\frac{a+b}{2}\right)f''(x)dx.\end{aligned}$$

By integration by parts, we deduce

$$\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)(f''(x)-k)dx = I_1 - k\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)dx$$
$$= I_1 - k\frac{(b-a)^2}{96} \tag{4}$$

and

$$\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)(K-f''(x))dx = K\frac{(b-a)^2}{96} - I_1. \tag{5}$$

Also, we have

$$\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b}(b-x)\left(x-\frac{a+b}{2}\right)(f''(x)-k)dx = I_2 - k\frac{(b-a)^2}{96}, \tag{6}$$

$$\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b}(b-x)\left(x-\frac{a+b}{2}\right)(K-f''(x))dx = K\frac{(b-a)^2}{96} - I_2. \tag{7}$$

By Chebychev integral inequality, we have the following inequalities:

$$\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)(f''(x)-k)dx$$
$$\leq \frac{1}{2(b-a)}\frac{\int_a^{\frac{a+b}{2}}(x-a)(f''(x)-k)dx\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)(f''(x)-k)dx}{\int_a^{\frac{a+b}{2}}(f''(x)-k)dx} \tag{8}$$

$$\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b}(b-x)\left(x-\frac{a+b}{2}\right)(f''(x)-k)dx$$
$$\leq \frac{1}{2(b-a)}\frac{\int_{\frac{a+b}{2}}^{b}(b-x)(f''(x)-k)dx\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)(f''(x)-k)dx}{\int_{\frac{a+b}{2}}^{b}(f''(x)-k)dx} \tag{9}$$

$$\frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}(x-a)\left(\frac{a+b}{2}-x\right)(K-f''(x))dx$$
$$\leq \frac{1}{2(b-a)}\frac{\int_a^{\frac{a+b}{2}}(x-a)(K-f''(x))dx\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)(K-f''(x))dx}{\int_a^{\frac{a+b}{2}}(K-f''(x))dx} \tag{10}$$

$$\frac{1}{2(b-a)}\int_{\frac{a+b}{2}}^{b}(b-x)\left(x-\frac{a+b}{2}\right)(K-f''(x))dx$$
$$\leq \frac{1}{2(b-a)}\frac{\int_{\frac{a+b}{2}}^{b}(b-x)(K-f''(x))dx\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)(K-f''(x))dx}{\int_{\frac{a+b}{2}}^{b}(K-f''(x))dx}. \tag{11}$$

By integration by parts, we have the following equalities:

$$\int_a^{\frac{a+b}{2}}(x-a)(f''(x)-k)dx$$

$$= \left[f'\left(\frac{a+b}{2}\right)\right] - k\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) - \left(f\left(\frac{a+b}{2}\right) - f(a) + k\left(\frac{(a+b)^2}{8} - \frac{a^2}{2}\right)\right) = \alpha_1 \tag{12}$$

$$\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)(f''(x) - k)\,dx$$
$$= (ka - f'(a))\left(\frac{b-a}{2}\right) + \left[f\left(\frac{a+b}{2}\right) - f(a) - k\left(\frac{(a+b)^2}{8} - \frac{a^2}{2}\right)\right] = \alpha_2 \tag{13}$$

$$\int_a^{\frac{a+b}{2}} (f''(x) - k)\,dx = f'\left(\frac{a+b}{2}\right) - f'(a) - k\left(\frac{b-a}{2}\right) = \alpha_3 \tag{14}$$

$$\int_{\frac{a+b}{2}}^{b} (b-x)(f''(x) - k)\,dx$$
$$= \left[k\left(\frac{a+b}{2}\right) - f'\left(\frac{a+b}{2}\right)\right]\left(\frac{b-a}{2}\right) + f(b) - f\left(\frac{a+b}{2}\right) - k\left(\frac{b^2}{2} - \frac{(a+b)^2}{8}\right) = \alpha_4 \tag{15}$$

$$\int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right)(f''(x) - k)\,dx$$
$$= (f'(b) - kb)\left(\frac{b-a}{2}\right) - \left[f(b) - f\left(\frac{a+b}{2}\right) - k\left(\frac{(a+b)^2}{8} - \frac{b^2}{2}\right)\right] = \alpha_5 \tag{16}$$

$$\int_{\frac{a+b}{2}}^{b} (f''(x) - k)\,dx = f'(b) - f'\left(\frac{a+b}{2}\right) - k\left(\frac{b-a}{2}\right) = \alpha_6 \tag{17}$$

$$\int_a^{\frac{a+b}{2}} (x-a)(K - f''(x))\,dx$$
$$= -\left[f'\left(\frac{a+b}{2}\right)\right] - K\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right) + \left(f\left(\frac{a+b}{2}\right) - f(a) + K\left(\frac{(a+b)^2}{8} - \frac{a^2}{2}\right)\right) = \sigma_1 \tag{18}$$

$$\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)(K - f''(x))\,dx$$
$$= (f'(a) - Ka)\left(\frac{b-a}{2}\right) - \left[f\left(\frac{a+b}{2}\right) - f(a) - K\left(\frac{(a+b)^2}{8} - \frac{a^2}{2}\right)\right] = \sigma_2 \tag{19}$$

$$\int_a^{\frac{a+b}{2}} (K - f''(x))\,dx = f'(a) - f'\left(\frac{a+b}{2}\right) + K\left(\frac{b-a}{2}\right) = \sigma_3 \tag{20}$$

$$\int_{\frac{a+b}{2}}^{b} (b-x)(K - f''(x))\,dx$$
$$= \left[f'\left(\frac{a+b}{2}\right) - K\left(\frac{a+b}{2}\right)\right]\left(\frac{b-a}{2}\right) - f(b) + f\left(\frac{a+b}{2}\right) + K\left(\frac{b^2}{2} - \frac{(a+b)^2}{8}\right) = \sigma_4 \tag{21}$$

$$\int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right)(K - f''(x))\,dx$$
$$= (Kb - f'(b))\left(\frac{b-a}{2}\right) + \left[f(b) - f\left(\frac{a+b}{2}\right) - k\left(\frac{(a+b)^2}{8} - \frac{b^2}{2}\right)\right] = \sigma_5 \tag{22}$$

$$\int_{\frac{a+b}{2}}^{b} (K - f''(x))\,dx = f'\left(\frac{a+b}{2}\right) - f'(b) + K\left(\frac{b-a}{2}\right) = \sigma_6. \tag{23}$$

Substituting equalities (12), (13), (14) in inequality (8) and from (4), we have

$$I_1 \le k\frac{(b-a)^2}{96} + \frac{1}{2(b-a)}\Big(\frac{\alpha_1\alpha_2}{\alpha_3}\Big). \tag{24}$$

Substituting equalities (15), (16), (17) in inequality (9) and from (6), we have

$$I_2 \le k\frac{(b-a)^2}{96} + \frac{1}{2(b-a)}\Big(\frac{\alpha_4\alpha_5}{\alpha_6}\Big). \tag{25}$$

Adding (24) and (25), we obtain

$$I_1 + I_2 \le k\frac{(b-a)^2}{48} + \frac{1}{2(b-a)}\Big(\frac{\alpha_1\alpha_2}{\alpha_3} + \frac{\alpha_4\alpha_5}{\alpha_6}\Big). \tag{26}$$

Substituting equalities (18), (19), (20) in inequality (10) and from (5), we have

$$I_1 \ge K\frac{(b-a)^2}{96} - \frac{1}{2(b-a)}\Big(\frac{\sigma_1\sigma_2}{\sigma_3}\Big). \tag{27}$$

Substituting equalities (21), (22), (23) in inequality (11) and from (7), we have

$$I_2 \ge K\frac{(b-a)^2}{96} - \frac{1}{2(b-a)}\Big(\frac{\sigma_4\sigma_5}{\sigma_6}\Big). \tag{28}$$

Adding (27) and (28), we obtain

$$I_1 + I_2 \ge K\frac{(b-a)^2}{48} - \frac{1}{2(b-a)}\Big(\frac{\sigma_1\sigma_2}{\sigma_3} + \frac{\sigma_4\sigma_5}{\sigma_6}\Big). \tag{29}$$

From (26) and (29), we have,

$$K\frac{(b-a)^2}{48} - \frac{1}{2(b-a)}\Big(\frac{\sigma_1\sigma_2}{\sigma_3} + \frac{\sigma_4\sigma_5}{\sigma_6}\Big) \le I_1 + I_2 \le k\frac{(b-a)^2}{48} + \frac{1}{2(b-a)}\Big(\frac{\alpha_1\alpha_2}{\alpha_3} + \frac{\alpha_4\alpha_5}{\alpha_6}\Big)$$

where,

$$I_1 + I_2 = \frac{1}{2}\Big[f\Big(\frac{a+b}{2}\Big) + \frac{f(a)+f(b)}{2}\Big] - \frac{1}{b-a}\int_a^b f(x)\,dx.$$

And the theorem is proved. □

3. Applications to special means

We shall consider the means for arbitrary real numbers α, β, $\alpha \ne \beta$. We take

$$H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in R\backslash\{0\}, \qquad \text{(harmonic mean)}$$

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in R, \qquad \text{(arithmetic mean)}$$

$$L_n(\alpha,\beta) = \Big[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\Big]^{1/n}, n \in Z\backslash\{-1,0\},\ \alpha,\beta \in R,\ \alpha \ne \beta, \qquad \text{(generalized log-mean).}$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let* $0 < a < b$ *and* $n \in Z - \{-1,0\}$. *Then we have the inequality,*

$$\frac{(b-a)^2}{32} n(n+1)\Big[A\Big(\Big(\frac{3a+b}{4}\Big)^{-n-2}, \Big(\frac{a+3b}{4}\Big)^{-n-2}\Big)\Big] \leq N$$
$$\leq \frac{(b-a)^2}{64} n(n+1)\Big[A\Big(\Big(\frac{3a+b}{4}\Big)^{-n-2}, \Big(\frac{a+3b}{4}\Big)^{-n-2}\Big)\Big]$$

where $N = \frac{1}{2}[A^{-n}(a,b) + A(a^{-n}, b^{-n})] - L_{-n}^{-n}(a,b)$.

Proof. The assertion follows from Theorem 1 applied for $f(x) = x^{-n}$, $n \in Z - \{-1,0\}$. □

Proposition 2. *Let* $0 < a < b$ *and* $n \in Z - \{-1,0\}$. *Then we have the inequality, for all* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\Big|L_{-n}^{-n}(a,b) - \frac{1}{2}[A^{-n}(a,b) + A(a^{-n}, b^{-n})]\Big| \leq \frac{(b-a)^2}{2^{(2p+1)/p}} |n(n+1)| [B(p+1,p+1)]^{1/p} L_{-n-q}^{\frac{-n-q}{q}}(a,b)$$

where $B(p,q)$ *is Euler's Beta function.*

Proof. If we apply Theorem 2 for $f(x) = x^{-n}$ on $[a,b]$, we obtain that

$$\Big|\frac{1}{2}[A^{-n}(a,b) + A(a^{-n}, b^{-n})] - L_{-n}^{-n}(a,b)\Big|$$
$$\leq \frac{(b-a)^{(p+1)/p}}{2^{(2p+1)/p}} [B(p+1,p+1)]^{1/p} \Big(\int_a^b [-n(-n-1)x^{-n-2}]^q dx\Big)^{1/q}.$$

Using the facts that

$$\int_a^b x^{-nq-2q} dx = \frac{b^{-nq-2q+1} - a^{-nq-2q+1}}{-nq-2q+1} = \frac{b^{-n-q+1} - a^{-n-q+1}}{-n-q+1} = L_{-n-q}^{-n-q}(a,b)(b-a)$$

we find that

$$0 \leq \frac{1}{2}[A^{-n}(a,b) + A(a^{-n}, b^{-n})] - L_{-n}^{-n}(a,b)$$
$$\leq \frac{(b-a)^{(p+1)/p}}{2^{(2p+1)/p}} n(n+1)[B(p+1,p+1)]^{1/p} L_{-n-q}^{\frac{-n-q}{q}}(a,b)(b-a)^{1/q}$$
$$\leq \frac{(b-a)^2}{2^{(2p+1)/p}} n(n+1)[B(p+1,p+1)]^{1/p} L_{-n-q}^{\frac{-n-q}{q}}(a,b).$$

Hence, we have the conclusion. □

Proposition 3. *Let* $0 < a < b$. *Then we have the inequality*

$$\Big|L_{-2}^{-2}(a,b) - \frac{1}{2}[A^{-2}(a,b) + A(a^{-2}, b^{-2})] - \frac{1}{24}(b-a)H^{-1}(a^3, -b^3)\Big| \leq \frac{3}{32}(b-a)^2 H^{-1}(a^4, -b^4).$$

Proof. If we apply Theorem 3 for the convex mapping $f(x)=\frac{1}{x^2}$ and from the equality $f''(x)=\frac{6}{x^4}$, we have

$$\gamma=\frac{6}{\left(\frac{a+b}{2}\right)^4}\leq f''(x)\leq\frac{6}{a^4}=\mu,\qquad x\in\left[a,\frac{a+b}{2}\right)$$

and

$$\gamma'=\frac{6}{b^4}\leq f''(x)\leq\frac{6}{\left(\frac{a+b}{2}\right)^4}=\mu'=\gamma,\qquad x\in\left[\frac{a+b}{2},b\right].$$

By inequality (3), we obtain,

$$\left|L_{-2}^{-2}(a,b)-\frac{1}{2}[A^{-2}(a,b)+A(a^{-2},b^{-2})]-\frac{1}{24}(b-a)H^{-1}(a^3,-b^3)\right|\leq\frac{3}{32}(b-a)^2H^{-1}(a^4,-b^4).$$

Hence, we have the conclusion. □

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