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HOMOMORPHISM THEOREMS IN SUBTRACTION ALGEBRA

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Abstract. In this paper, we give homomorphism theorem between two subtraction algebras and investigate some related properties.

1. Introduction

B. M. Schein [8] considered systems of the form $(\Phi, \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun, H. S. Kim and E. H. Roh [4] introduced the notion of ideal in subtraction algebras and discussed characterization of ideals. In [5], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. In [2], Y. Ceven and M. A. Ozturk introduced some additional concepts on subtraction algebras, so called subalgebra, bounded subtraction algebra and union of subtraction algebras and investigated some related properties. Y. Çeven and Ş. Küçükkoç [3] introduced quotient subtraction algebras, and investigated some properties.

In this paper, we give homomorphism theorem between two subtraction algebras and investigate some related properties.

2. Preliminaries

An algebra $(X; -)$ with a single binary operation "-" is called subtraction algebra if for all $x, y, z \in X$ the following conditions hold:

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$(S1)$ $x-(y-x)=x$,

$(S2)$ $x-(x-y)=y-(y-x)$,

$(S3)$ $(x-y)-z=(x-z)-y$.

The subtraction determines an order relation on X as the following:

$$a\leq b\Leftrightarrow a-b=0$$

$0=a-a$ is an element of X and this property does not depend on the choice of $a\in X$. The ordered set $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to induced order. Here $a\wedge b=a-(a-b)$ and the complement of an element $b\in[0,a]$ is $a-b$.

In a subtraction algebra, the following are true [4], [7];

$(a1)$ $(x-y)-y=x-y$,

$(a2)$ $x-0=x$ and $0-x=0$,

$(a3)$ $(x-y)-x=0$,

$(a4)$ $x-(x-y)\leq y$,

$(a5)$ $(x-y)-(y-x)=x-y$,

$(a6)$ $x-(x-(x-y))=x-y$,

$(a7)$ $(x-y)-(z-y)\leq x-z$,

$(a8)$ $x\leq y$ if and only if $x=y-w$ for some $w\in X$,

$(a9)$ $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$,

$(a10)$ $x,y\leq z$ implies $x-y=x\wedge(z-y)$,

$(a11)$ $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$,

$(a12)$ $(x-y)-z=(x-z)-(y-z)$.

Definition 1 ([4])**.** A nonempty subset A of a sutraction algebra X is called an ideal of X if it satisfies, for all $x,y,z\in X$,

(1) $0\in A$,

(2) $(\forall x\in X)(\forall y\in A)(x-y\in A\Rightarrow x\in A)$.

Lemma 1 ([4])**.** *An ideal A of a subtraction algebra X has the following property:*

$(\forall x\in X)(\forall y\in A)(x\leq y\Rightarrow x\in A)$

Definition 2 ([6])**.** Let X be a subtraction algebra. For any $a,b\in X$, let $G(a,b)=\{x\in X: x-a\leq b\}$. X is said to be complicated if for any $a,b\in X$, the set $G(a,b)$ has the greatest element.

Note that $0,a,b\in G(a,b)$. The greatest element of $G(a,b)$ is denoted by $a+b$.

Proposition 1 ([6]). *If X is a c-subtraction algebra, then for all $x, y \in X$,*

(i) $x \leq x + y$, $y \leq x + y$,

(ii) $x + 0 = x = 0 + x$,

(iii) $x + y = y + x$,

(iv) $x \leq y \Rightarrow x + z \leq y + z$,

(v) $x \leq y \Rightarrow x + y = y$,

(vi) $x + y$ *is the least upper bound of* x *and* y.

Lemma 2 ([6]). *If X is a c-subtraction algebra and $x, y, z \in X$, then*

$$(x + y) + z = x + (y + z).$$

Definition 3 ([2]). Let X be a subtraction algebra and I be a nonempty subset of X. If $x - y \in I$, for all $x, y \in I$, then I is called a subalgebra of X.

Theorem 1 ([2]). *Any ideal of a subtraction algebra X is subalgebra of X.*

Let X be a subtraction algebra and I be an ideal of X. The relation "$\sim$" as following is an equivalence relation on X,

$$x \sim y \Leftrightarrow x - y \in I \text{ and } y - x \in I \tag{2.1}$$

Denote the equivalence class containing x by $\bar{x} = \{y \in X : y \sim x\}$.

Lemma 3 ([3]). *Let X be a subtraction algebra and I be an ideal of X. The relation "$\sim$" as in* (2.1) *is congruence relation on X.*

Proposition 2 ([3]). *Let X be a subtraction algebra and I be an ideal of X then $I = \bar{0}$.*

Definition 4 ([3]). Let X be a subtraction algebra and I be an ideal of X. Let "$\sim$" be an equivalence relation as in (2.1). The set of all equivalence classes in X is denoted by X/I and called the quotient set of X by I.

Theorem 2 ([3]). *Let X be a subtraction algebra and I be an ideal of X. Then X/I is a subtraction algebra with the operation "$-$" given by $\bar{x} - \bar{y} = \overline{x - y}$ for all $x, y \in X$.*

Theorem 3 ([3]). *If I and J are any two ideals of subtraction algebra X and $I \subset J$, then the following conditions are satisfied:*

(*a*) *I is an ideal of subalgebra J,*

(*b*) *J/I is an ideal of quotient algebra of X/I.*

Definition 5 ([3]). Let $(X,-)$ and $(Y,\ominus)$ be any two subtraction algebras. If the mapping $\varphi : X \to Y$, for all x, y in X, satisfies the condition

$$\varphi(x-y) = \varphi(x) \ominus \varphi(y)$$

then φ is called a homomorphism.

Theorem 4 ([3]). *If I is an ideal of subtraction algebra X.The mapping φ is from X to X/I which is given by $\varphi(x) = \overline{x}$ for all x in X is a homomorphism.*

Lemma 4. *If X is a subtraction algebra and $x, y, z \in X$, then*

$$(x-y)-(x-z) \leq z-y.$$

Proof. By $(S3)$ and $(S2)$,

$$\begin{aligned}(x-y)-(x-z) &= (x-(x-z))-y \\ &= (z-(z-x))-y \\ &= (z-y)-(z-x).\end{aligned}$$

Hence,

$$\begin{aligned}((x-y)-(x-z))-(z-y) &= ((z-y)-(z-x))-(z-y) \\ &= ((z-y)-(z-y))-(z-x) \\ &= 0-(z-x) = 0.\end{aligned}$$

Therefore, we have

$$(x-y)-(x-z) \leq z-y.$$ □

Lemma 5. *Let X be a c-subtraction algebra. Then,for all $x, y, z \in X$,*

$$(x-y)-z = x-(y+z).$$

Proof. By $(a7)$ and $(S3)$,

$$\begin{aligned}(x-((x-y)-z))-z &= (x-z)-((x-y)-z) \\ &\leq x-(x-y) \leq y.\end{aligned}$$

Hence

$$x-((x-y)-z) \leq y+z.$$

By $(S3)$,

$$x-(y+z) \leq (x-y)-z.$$

On the other hand, by Lemma 4,

$$(x-y)-(x-(y+z)) \leq (y+z)-y \leq z,$$

and so

$$(x-y)-z \leq x-(y+z)$$

That is, $(x-y)-z = x-(y+z)$. □

Lemma 6. *Let X be a c-subtraction algebra. Then,for all $x, y, z \in X$,*

$$(x+z)-(y+z) \leq x-y \leq x+y.$$

Proof. Using $(a4)$ and definition of $+$, we get

$$x \leq (x-y)+y.$$

By Proposition-1(iv) and Lemma 2,

$$x+z \leq ((x-y)+y)+z = (x-y)+(y+z).$$

Thus,

$$(x+z)-(y+z) \leq ((x-y)+(y+z))-(y+z) \leq x-y.$$

On the other hand, we have

$$x-y \leq x = x+0 \leq x+y.$$

Therefore, $(x+z)-(y+z) \leq x+y$. □

Lemma 7. *Let X be a c-subtraction algebra. Then,for all $x, y, z \in X$,*

$$(x+y)-z = (x-z)+(y-z).$$

Proof. By Lemma 6,

$$(x+y)-(y+z) = (x+y)-(z+y) \leq x-z.$$

Using $(S3)$, we have

$$(x+y)-(x-z) \leq y+z.$$

Then

$$\begin{aligned}((x+y)-z)-(x-z) &= ((x+y)-(x-z))-z \\ &\leq (y+z)-z = (y+z)-(z+z) \leq y-z,\end{aligned}$$

so,

$$(x+y)-z \leq (x-z)+(y-z).$$

On the other hand, since $x, y \leq x+y$, it follows that

$$x-z \leq (x+y)-z,\ y-z \leq (x+y)-z.$$

Therefore,

$$\begin{aligned}(x-z)+(y-z) &\leq ((x+y)-z)+(y-z)\\ &\leq ((x+y)-z)+((x+y)-z)\\ &= (x+y)-z.\end{aligned}$$

That is, $(x+y)-z=(x-z)+(y-z)$. □

3. Homomorphisms and isomorphisms

Suppose that $(X,-)$ and $(Y,\ominus)$ are two subtraction algebras and $f: X \to Y$ is a homomorphism. If in addition, the mapping f is onto, i. e., $f(X)=Y$, where $f(X)=\{f(x): x \in X\}$, then f is called an epimorphism and Y is said to be homomorphic image of X. The mapping is called an isomorphism if it is both an epimorphism and one-to-one. If there exists an isomorphism $f: X \to Y$ then we call X to be isomorphic to Y, written $X \cong Y$. Obviously, $X \cong Y$ implies $Y \cong X$ and $X \cong Y$, $Y \cong Z$ implies $X \cong Z$. In case $X=Y$ a homomorphism is called an endomorphism and an isomorphism is referred as an automorphism. The identity map $1: X \to X$ is clearly an endomorphism. So that $X \cong X$.

The set of all homomorphisms from X to Y is denoted by $Hom(X,Y)$. This set is always nonempty since it contains the zero homomorphism $0: X \to Y$ which sends every element of X to $0'$. For any $f \in Hom(X,Y)$ and any nonempty subset $B \subseteq X$, the set $\{x \in X: f(x) \in B\}$ is denoted by $f^{-1}(B)$, called the inverse image of B under f. In particular, $f^{-1}(\{0'\})$ is called kernel of f. Note that

$$f^{-1}(\{0'\})=\{x \in X: f(x)=0'\}$$

We will simply write $Ker(f)$ instead of $f^{-1}(\{0'\})$.

Definition 6. Let X and Y be two subtraction algebras and $f: X \to Y$ be a mapping. If for any $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$ then f is called isotone.

Theorem 5. *Let $(X,-)$ and $(Y,\ominus)$ be two subtraction algebras and $f: X \to Y$ be a homomorphism. Then,*

(a) $f(0)=0'$,

(b) *f is isotone.*

Proof. (a) $f(0) = f(0-0) = f(0) \ominus f(0) = 0'$ since Y be a subtraction algebra.

(b) If $x, y \in X$ and $x \le y$ then $x - y = 0$, by (a), $f(x) \ominus f(y) = f(x-y) = f(0) = 0'$. Hence $f(x) \le f(y)$. □

Theorem 6. *Let $(X,-)$ and $(Y,\ominus)$ be two subtraction algebras and let B be an ideal of Y. Then for any $f \in Hom(X,Y)$, $f^{-1}(B)$ is an ideal of X.*

Proof. From Theorem 5, $0 \in f^{-1}(B)$. Assume that $x - y \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(x) \ominus f(y) \in B$ and $f(y) \in B$. Since B is an ideal of Y, $f(x) \in B$. Thus, $x \in f^{-1}(B)$. Hence, $f^{-1}(B)$ is an ideal of X. □

Since $\{0'\}$ is an ideal of Y, we have $Kerf$ is an ideal of X.

Example 1. From Theorem 4, quotient algebra X/I is a homomorphic image of X.

Theorem 7. *Let $(X,-)$ and $(Y,\ominus)$ be two subtraction algebras and $f : X \to Y$ be an epimorphism. Then, $X/Kerf \cong Y$.*

Proof. Since $Kerf$ is an ideal of X, by Theorem 6, $X/Kerf$ is an subtraction algebra. Assume $g : X/Kerf \to Y$ such that $g(\bar{x}) = f(x)$. If $\bar{x} = \bar{y}$ then, $x - y$, $y - x \in Kerf$, so

$$f(x-y) = f(x) \ominus f(y) = 0'$$
$$f(y-x) = f(y) \ominus f(x) = 0'$$

By $(a2)$ we have $f(x) = f(y)$, i. e., $g(\bar{x}) = g(\bar{y})$. Hence g is well-defined. For any $y \in Y$, there exists $x \in X$ such that $y = f(x)$ as f is onto. Hence $g(\bar{x}) = y$, which means that $g : X/Kerf \to Y$ is onto.

If $\bar{x} \neq \bar{y}$, then $x - y \notin Kerf$ or $y - x \notin Kerf$. Suppose $x - y \notin Kerf$,

$$f(x) \ominus f(y) = f(x-y) \neq 0'$$

Therefore $f(x) \neq f(y)$. That is, g one-to-one. Since

$$g(\bar{x} - \bar{y}) = g(\overline{x-y}) = f(x-y) = f(x) \ominus f(y)$$
$$= g(\bar{x}) \ominus g(\bar{y}),$$

g is a homomorphism.

Therefore, $X/Kerf \cong Y$. □

Theorem 8. *Let $(X,-)$ and $(Y,\ominus)$ be two subtraction algebras and $f : X \to Y$ be an epimorphism. If J is ideal of Y, then $X/I \cong Y/J$ where $I = f^{-1}(J)$.*

Proof. From Theorem 4, there is, $\mu : Y \to Y/J$ epimorphism. Then $\mu \circ f : X \to Y/J$ is an epimorphism. We now prove that $Ker(\mu \circ f) = f^{-1}(J)$. For any $x \in X$, we have

$$(\mu \circ f)(x) = \mu(f(x)) = \overline{f(x)}$$

Suppose $y \in f^{-1}(J)$. Then $f(y) \in J$, and so $\overline{f(y)} = J$. That is, $(\mu \circ f)(y) = J$. Hence $y \in Ker(\mu \circ f)$. Thus, we have $f^{-1}(J) \subseteq Ker(\mu \circ f)$. Inverse, we assume $x \in Ker(\mu \circ f)$, i. e., $(\mu \circ f)(x) = J$. Therefore, we have $\overline{f(y)} = J$, and so $f(x) \in J$, i. e., $x \in f^{-1}(J)$. Thus, by Theorem 7, $X/I \cong Y/J$. □

Lemma 8. *Let $(X, -)$ and $(Y, \ominus)$ be two subtraction algebras, $f : X \to Y$ be an epimorphism and I be an ideal of X. If $K\text{erf} \subseteq I$, then $f^{-1}(f(I)) = I$.*

Proof. Obviously, $I \subseteq f^{-1}(f(I))$. Inverse, we assume $x \in f^{-1}(f(I))$, then $f(x) \in f(I)$. There exists $y \in I$ such that $f(x) = f(y)$, so

$$f(x - y) = f(x) \ominus f(y) = 0',$$

hence $x - y \in K\text{erf} \subseteq I$. Since $x - y \in I$, $y \in I$, we have $x \in I$.Therefore $f^{-1}(f(I)) \subseteq I$. □

Theorem 9. *Let I and J be two ideals of a subtraction algebra X, $I \subseteq J$ and $f : X \to X/I$ and $g : X/I \to (X/I)/(J/I)$ be epimorphisms. Then $X/J \cong (X/I)/(J/I)$.*

Proof. Let $h = g \circ f$. Then $h : X \to (X/I)/(J/I)$ is an epimorphism. Hence

$$X/Kerh \cong (X/I)/(J/I)$$

We must prove that $Kerh = J$. Since

$$Kerh = \{x \in X : h(x) = J/I\},$$

by Theorem 3, Lemma 8 and Theorem 8 we have

$$Kerh = h^{-1}(h(J)) = J$$

The proof is complete. □

Definition 7. Let X be a subtraction algebra. For a fixed $a \in X$, we define a map $T_a : X \to X$ such that $T_a(x) = x - a$ for all $x \in X$ and call T_a right map on X. A left map L_a is defined by a similar way.

Note that for all $x \in X$,

$(T_a \circ T_a)(x) = T_a(T_a(x)) = T_a(x-a) = x-(x-a) = x-a = T_a(x)$ by $(a1)$, and

$[(L_a \circ L_a) \circ L_a](x) = (L_a \circ L_a)(a-x) = L_a(a-(a-x)) = a-(a-(a-x)) = a-x = L_a(x)$ by $(a6)$.

Furthermore, for all $x, y \in X$,

$$T_a\left(x-y\right) = \left(x-y\right)-a = (x-a)-\left(y-a\right) = T_a(x)-T_a\left(y\right)$$

by $(a12)$, this says T_a is a homomorphism.

L_a is a homomorphism if and only if $a = 0$. Assume that L_a is a homomorphism. $L_a(0) = L_a(0-0) = L_a(0)-L_a(0) = 0$. That is, $a = a-0 = 0$. If $a = 0$, then $L_a\left(x-y\right) = a-\left(x-y\right) = 0-\left(x-y\right) = 0 = (0-x)-\left(0-y\right) = L_a(x)-L_a\left(y\right)$.

Theorem 10. *Let a be a fixed element of a subtraction algebra X. Then the following are equivalent;*

(a) L_a *is onto,*

(b) L_a *is one-to-one,*

(c) $x = a-(a-x)$, *for all* $x \in X$.

Proof. $(a) \Rightarrow (b)$ Assume (a) and $L_a(x) = L_a\left(y\right)$, then $a-x = a-y$. For this element x, since L_a is onto, there exists $z \in X$ such that $a-z = x$, hence $a-y = a-x = a-(a-z) \leq z$, by $(a4)$ and so $\left(a-y\right)-z = 0$. It follows that $x-y = (a-z)-y = \left(a-y\right)-z = 0$, by $(S3)$. In a similar way we have $y-x = 0$. This implies $x = y$, i.e., L_a is one-to-one.

$(b) \Rightarrow (c)$ Let L_a be one-to-one. Since $(L_a \circ L_a) \circ L_a = L_a$, for all $x \in X$, $L_a(a-(a-x)) = L_a(x)$ and so $x = a-(a-x)$.

$(c) \Rightarrow (a)$ Since for any $x \in X$, $x = a-(a-x)$, $L_a(a-x) = x$. Hence L_a is onto. □

Let $IL(X)$ be set of all left maps an X. We defined an operation $\ominus$ in $IL(X)$ as follows. For any $L_a, L_b \in IL(X)$ and for all $x \in X$,

$$(L_a \ominus L_b)(x) = L_a(x)-L_b(x)$$

Note that,

$$\begin{aligned} L_a(x)-L_b(x) &= (a-x)-(b-x) \\ &= (a-b)-x = L_{a-b}(x) \end{aligned}$$

by $(a12)$, and so $L_a \ominus L_b \in IL(X)$.

Theorem 11. *Let X be a subtraction algebra. Then X is isomorphic to $IL(X)$.*

Proof. First, we show that $(IL(X), \ominus)$ is a subtraction algebra. Indeed, $L_a \ominus (L_b \ominus L_a) = L_a \ominus L_{b-a} = L_{a-(b-a)} = L_a$, by $(S1)$. $L_a \ominus (L_a \ominus L_b) = L_b \ominus (L_b \ominus L_a)$, by $(S2)$. $(L_a \ominus L_b) \ominus L_c = (L_a \ominus L_c) \ominus L_b$.

Now, we show that a map $f : X \to IL(X)$, $f(x) = L_x$ is an isomorphism.

Suppose that, $f(x) = f(y)$, that is, $L_x = L_y$ and so for all $z \in X$, $L_x(z) = L_y(z)$ and hence $x - z = y - z$. If we take $z = y$, $x - y = y - y = 0$. If we take $z = x$, $y - x = x - x = 0$. Therefore, $x = y$, that is, f one-to-one. Clearly, f is also onto. For all $z \in X$,

$$\begin{aligned} f(x-y)(z) &= L_{x-y}(z) = (x-y) - z \\ &= (x-z) - (y-z) = L_x(z) - L_y(z) \\ &= (L_x \ominus L_y)(z) \end{aligned}$$

it follows that f is a homomorphism. And so $X \cong IL(X)$. □

Theorem 12. *Let X and Y be c-subtraction algebras and $f \in Hom(X, Y)$. Then for all $x, y \in X$, $f(x+y) = f(x) + f(y)$, i. e., f is a homomorphism with respect to $+$.*

Proof. For any $x, y \in X$, $x + y \in G(x, y)$. Since $f \in Hom(X, Y)$, f is isotone. Thus,

$$\begin{aligned} (x+y) - x &\le y \\ f(x+y) - f(x) &\le f(y) \end{aligned}$$

and so $f(x+y) \in G(f(x), f(y))$. Since $f(x) + f(y)$ is the greatest element of $G(f(x), f(y))$, we have $f(x+y) \le f(x) + f(y)$.

On the other hand, $x \le x + y$ and $y \le x + y$. Therefore, $f(x) \le f(x+y)$, $f(y) \le f(x+y)$. Since $f(x) + f(y)$ is the least upper bound of $f(x)$ and $f(y)$, we have $f(x) + f(y) \le f(x+y)$. That is, $f(x+y) = f(x) + f(y)$. □

Theorem 13. *Let X be a c-subtraction algebra. Then for every $a \in X$, right map T_a is an endomorphism on $+$.*

Proof. For any $x, y \in X$,

$$\begin{aligned} T_a(x+y) &= (x+y) - a = (x-a) + (y-a) \\ &= T_a(x) + T_a(y) \end{aligned}$$

by Lemma 7. □

Let $IR(X)$ be set of all right maps on X. We defined an operation $\circ$ in $IR(X)$ as follows: For any $T_a, T_b \in IR(X)$ and for any $x \in X$,

$$(T_a \circ T_b)(x) = T_a(T_b(x)).$$

Note that, for any $T_a, T_b \in IR(X)$, $T_a \circ T_b = T_{a+b}$. Indeed, for any $x \in X$,

$$\begin{aligned}(T_a \circ T_b)(x) &= T_a(T_b(x)) = T_a(x-b)\\ &= (x-b)-a = (x-a)-b\\ &= x-(a+b) = T_{a+b}(x),\end{aligned}$$

by Lemma 5.

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