CROSSCUTS IN SPHERE

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Abstract. Crosscut property of subsets in the unit sphere $S^n \subset E^{n+1}$ has been defined. Its relation with convexity has been studied. Illustrating examples are given.

The crosscut of a subset in Euclidean space $E^n$ has been considered in [4, 5] as follows: A crosscut of a set $A \subset E^n$ is a closed segment $[xy]$ such that the open segment $(xy) = [xy] \setminus \{x, y\}$ is contained in $Int(A)$ and $x, y \in \partial A$. The following result relating this concept with convexity is proved in [4, 5].

Theorem 1. If an open set $K \subset E^n$ has no crosscuts, then its complement $K^c$ is a convex set.

As far as I am concerned no studies have been established about the same concept in $S^n$ as an ambient space. Consequently, we deal throughout this article with this subject.

Let $A \subset S^n$ be a subset. $A$ is convex if for each pair of points there exists a unique minimal geodesic segment $[pq]$ joining $p$ and $q$ such that $[pq] \subset A$. If in addition $\partial A$ does not contain any geodesic segments, then $A$ is called strictly convex.

For the pair of points $p, q \in S^n$ if there exists a unique minimal closed geodesic segment joining $p$ and $q$ it will be denoted by $[pq]$. Moreover, $(pq) = [pq] \setminus \{p, q\}$ will denote the open geodesic segment from $p$ to $q$. The geodesic segment always exists if and only if $p, q \in S^n$ are non-antipodal points.

Definition 2. Let $A \subset S^n$ be a subset. $A$ has a crosscut with respect to the boundary points $p, q \in \partial A$ if there exists a unique minimal closed geodesic segment $[pq]$ joining $p$ and $q$ such that $(pq) \subset Int(A)$.

In the light of this definition the boundary points $p, q$ should be antipodal points.

Definition 3. A subset $A \subset S^n$ has the crosscut property if $A$ has a crosscut for each pair of boundary points.

It is easy to show that every open geodesic ball with center $p$ and radius $r < \pi/2$ has as a convex body in $S^n$—the crosscut property. If $r = \pi/2$ we obtain an open hemisphere (also a convex body) with no crosscuts.

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From this argument we have $B(p, \pi/2)$ as an open subset of $S^n$ with no crosscuts while the closed hemisphere $B(p, \pi/2)^c$ is a non-convex subset of $S^n$. Hence Theorem 1 is not valid generally in $S^n$. The following result which represents the main one of this work is a modified form of Theorem 1 in $S^n$.

**Theorem 4.** Let $A$ be an open subset of $S^n$ whose boundary $\partial A$ is free from antipodal points. If $A$ has no crosscuts then $A^c$ is convex.

**Proof.** Assume in contrary that the closed subset $A^c$ is not convex. Then we have to consider the following cases:

(a) There exists a non-antipodal pair of points such that $[pq] \not\subset A^c$. Hence, there exists a point $m \in [pq]$ belonging to $A$. This would imply that $A$ has a crosscut which is a subsegment of $[pq]$ contradicting the hypothesis.

(b) $A^c$ contains a pair of antipodal points $\omega_1, \omega_2$. As $\partial A$ is free from the antipodal points we may assume that $\omega_1 \not\in \partial A$ and $\omega_2 \not\in \partial A$ or $\omega_1 \in \partial A$ and $\omega_2 \not\in \partial A$. Consider the geodesic segment $\gamma$ joining $\omega_1$ and $\omega_2$ and passing through a point $m \in A$ (See Figure 1(a), 1(b)). Again we obtain a subsegment $[pq]$ or $[\omega_1q]$, $p, q \in \partial A$ of $\gamma$ which is a crosscut of $A$. This is a contradiction and the proof is complete.

![Figure 1](image)

**Remarks**

(i) Openness in Theorem 4 is so important as if we consider a small geodesic sphere $\partial B(p, r), r < \pi/2$, we have a closed subset of $S^n$ which does not have any crosscut while $S^n \setminus \partial B(p, r)$ is not a convex subset of $S^n$.

(ii) To show that the condition ”$\partial A$ is free from antipodal points” is essential in Theorem 4, consider $A = S^n \setminus \{p, q\}$ where $p, q$ are antipodal points. The subset $A$ is open with no crosscuts and $\partial A = \{p, q\}$, $A^c = \{p, q\}$. Clearly $A^c$ is non-convex.

**Theorem 5.** Let $A$ be an open subset of $S^n$ whose boundary $\partial A$ is contained in a small geodesic ball $B(p, \delta)$, $\delta < \pi/2$. If $A$ has no crosscuts, then $A^c$ is convex.

The proof is direct in the light of that of Theorem 4 as $\partial A$ here is also free from antipodal points.
Theorem 6. Let $A \subset S^n$ be a closed convex subset. Then $A^c$ does not have any crosscut.

Proof. The proof is direct as if we consider an arbitrary pair of boundary points $p, q \in \partial A$ there exists a unique minimal geodesic segment $[pq]$ joining $p$ and $q$ such that $[pq] \subset A$. Consequently, $(pq) \notin A^c$ and hence is not a crosscut of $A^c$.

Corollary 7. Let $A \subset S^n$ be a closed strictly convex subset. Then $A$ has the crosscut property.

Theorem 8. Let $A \subset S^n$ be a closed subset satisfying the crosscut property. Then $A$ is a strictly convex subset.

Proof. Assume firstly that $A$ is a non-convex subset of $S^n$. Hence $A$ has a pair of points, say $p, q$, such that either:
(i) $p, q$ are antipodal points, or,
(ii) $p, q$ has a closed minimal segment $[pq]$ such that $[pq] \subset A$.

In case (i) if $p, q \in \partial A$, then $A$ does not have the crosscut property. Hence one of the points, say $p$, at least, should belong to $Int(A)$. In this case let $m \in A^c$ be an arbitrary point. The geodesic segment $\gamma_{pmq}$ joining $p$ and $q$ through $m$ will contain a unique minimal subsegment $[wq]$, $w, q \in \partial A$ which is not a crosscut of $A$ (See Figure 2). This is a contradiction. If $p$ and $q$ are both interior points of $A$ the proof goes similarly.

Case (ii) will also give-in the light of the above argument-that $A$ does not satisfy the crosscut property.

The proof of strict convexity of $A$ is direct.

Theorem 9. Let $M$ be an $n$-dimensional compact smooth manifold and $f : M \rightarrow S^{n+1}$ an imbedding such that $f(M)$ is a boundary of two open subsets of $S^{n+1}$ each with no crosscuts. Then $M$ is diffeomorphic to $S^n$ and $f(M)$ is geodesic hypersphere of $S^{n+1}$.

Proof. Let us consider an arbitrary point $p \in M$. Let $B(f(p), \varepsilon)$ be a sufficiently small convex geodesic ball $[1, 3]$. Let $f(q) \in f(M) \cap B(f(p), \varepsilon)$ be an arbitrary point. By hypotheses $[f(p)f(q)] \subset f(M) \cap B(f(p), \varepsilon)$. Lifting $f(M) \cap B(f(p), \varepsilon)$ using $\exp_{f(p)}^{-1}$ to
$T_{f(p)}S^{n+1}$ we have that the height function $h$ (and hence the second fundamental form at $p$) of $f(M) \cap B(f(p), \varepsilon)$ is the zero function $[1, 3]$. Hence the sectional curvature $K_p$ of $f(M)$ at $f(p)$ is 1. Similar argument shows that $K \equiv 1$ on $f(M)$. Finally, in [2], it is proved that $f(M)$ is a geodesic hypersphere and hence our result.

References


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