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CROSSCUTS IN SPHERE

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Abstract. Crosscut property of subsets in the unit sphere $S^n \subset E^{n+1}$ has been defined. Its relation with convexity has been studied. Illustrating examples are given.

The crosscut of a subset in Euclidean space E^n has been considered in [4, 5] as follows: A crosscut of a set $A \subset E^n$ is a closed segment $[xy]$ such that the open segment $(xy) = [xy] \setminus \{x, y\}$ is contained in $Int(A)$ and x, $y \in \partial A$. The following result relating this concept with convexity is proved in [4, 5].

Theorem 1. *If an open set $K \subset E^n$ has no crosscuts, then its complement K^c is a convex set.*

As far as I am concerned no studies have been established about the same concept in S^n as an ambient space. Consequently, we deal throughout this article with this subject.

Let $A \subset S^n$ be a subset. A is convex if for each pair of points there exists a unique minimal geodesic segment $[pq]$ joining p and q such that $[pq] \subset A$. If in addition ∂A does not contain any geodesic segments, then A is called *strictly convex*.

For the pair of points p, $q \in S^n$ if there exists a unique minimal closed geodesic segment joining p and q it will be denoted by $[pq]$. Moreover, $(pq) = [pq] \setminus \{p, q\}$ will denote the open geodesic segment from p to q. The geodesic segment always exists if and only if p, $q \in S^n$ are non-antipodal points.

Definition 2. Let $A \subset S^n$ be a subset. A has a crosscut with respect to the boundary points p, $q \in \partial A$ if there exists a unique minimal closed geodesic segment $[pq]$ joining p and q such that $(pq) \subset Int(A)$.

In the light of this definition the boundary points p, q should be antipodal points.

Definition 3. A subset $A \subset S^n$ has the crosscut property if A has a crosscut for each pair of boundary points.

It is easy to show that every open geodesic ball with center p and radius $r < \pi/2$ has-as a convex body in S^n—the crosscut property. If $r = \pi/2$ we obtain an open hemishpere (also a convex body) with no crosscuts.

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From this argument we have $B(p, \pi/2)$ as an open subset of S^n with no crosscuts while the closed hemisphere $B(p, \pi/2)^c$ is a non-convex subset of S^n. Hence Theorem 1 is not valid generally in S^n. The following result which represents the main one of this work is a modified form of Theorem 1 in S^n.

Theorem 4. *Let A be an open subset of S^n whose boundary ∂A is free from antipodal points. If A has no crosscuts then A^c is convex.*

Proof. Assume in contrary that the closed subset A^c is not convex. Then we have to consider the following cases:

(a) There exists a non-antipodal pair of points such that $[pq] \not\subset A^c$. Hence, there exists a point $m \in [pq]$ belonging to A. This would imply that A has a crosscut which is a subsegment of $[pq]$ contradicting the hypothesis.

(b) A^c contains a pair of antipodal points ω_1, ω_2. As ∂A is free from the antipodal points we may assume that $\omega_1 \notin \partial A$ and $\omega_2 \notin \partial A$ or $\omega_1 \in \partial A$ and $\omega_2 \notin \partial A$. Consider the geodesic segment γ joining ω_1 and ω_2 and passing through a point $m \in A$ (See Figure 1(a), 1(b)). Again we obtain a subsegment $[pq]$ or $[\omega_1 q]$, p, $q \in \partial A$ of γ which is a crosscut of A. This is a contradiction and the proof is complete.

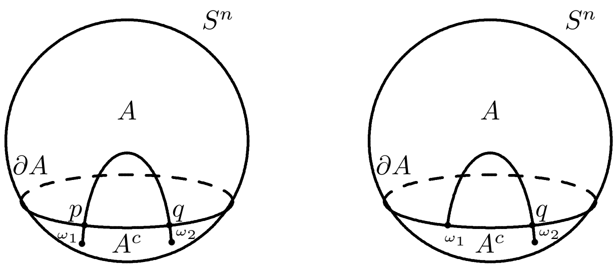

Figure 1.

Remarks

(i) Openness in Theorem 4 is so important as if we consider a small geodesic sphere $\partial B(p, r)$, $r < \pi/2$, we have a closed subset of S^n which does not have any crosscut while $S^n \setminus \partial B(p, r)$ is not a convex subset of S^n.

(ii) To show that the condition "∂A is free from antipodal points" is essential in Theorem 4, consider $A = S^n \setminus \{p, q\}$ where p, q are antipodal points. The subset A is open with no crosscuts and $\partial A = \{p, q\}$, $A^c = \{p, q\}$. Clearly A^c is non-convex.

Theorem 5. *Let A be an open subset of S^n whose boundary ∂A is contained in a small geodesic ball $B(p, \delta)$, $\delta < \pi/2$. If A has no crosscuts, then A^c is convex.*

The proof is direct in the light of that of Theorem 4 as ∂A here is also free from antipodal points.

Theorem 6. *Let $A \subset S^n$ be a closed convex subset. Then A^c does not have any crosscut.*

Proof. The proof is direct as if we consider an arbitrary pair of boundary points p, $q \in \partial A$ there exists a unique minimal geodesic segment $[pq]$ joining p and q such that $[pq] \subset A$. Consequently, $(pq) \not\subset A^c$ and hence is not a crosscut of A^c.

Corollary 7. *Let $A \subset S^n$ be a closed strictly convex subset. Then A has the crosscut property.*

Theorem 8. *Let $A \subset S^n$ be a closed subset satisfying the crosscut property. Then A is a strictly convex subset.*

Proof. Assume firstly that A is a non-convex subset of S^n. Hence A has a pair of points, say p, q, such that either:
(i) p, q are antipodal points, or,
(ii) p, q has a closed minimal segment $[pq]$ such that $[pq] \not\subset A$.

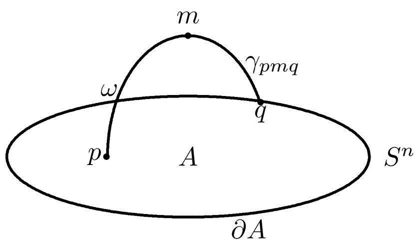

Figure 2.

In case (i) if p, $q \in \partial A$, then A does not have the crosscut property. Hence one of the points, say p, at least, should belong to $Int(A)$. In this case let $m \in A^c$ be an arbitrary point. The geodesic segment γ_{pmq} joining p and q through m will contain a unique minimal subsegment $[wq]$, w, $q \in \partial A$ which is not a crosscut of A (See Figure 2). This is a contradiction. If p and q are both interior points of A the proof goes similarly.

Case (ii) will also give-in the light of the above argument-that A does not satisfy the crosscut property.

The proof of strict convexity of A is direct.

Theorem 9. *Let M be an n-dimensional compact smooth manifold and $f : M \to S^{n+1}$ an imbedding such that $f(M)$ is a boundary of two open subsets of S^{n+1} each with no crosscuts. Then M is diffeomorphic to S^n and $f(M)$ is geodesic hypersphere of S^{n+1}.*

Proof. Let us consider an arbitrary point $p \in M$. Let $B(f(p), \varepsilon)$ be a sufficiently small convex geodesic ball [1, 3]. Let $f(q) \in f(M) \cap B(f(p), \varepsilon)$ be an arbitrary point. By hypotheses $[f(p)f(q)] \subset f(M) \cap B(f(p), \varepsilon)$. Lifting $f(M) \cap B(f(p), \varepsilon)$ using $\exp^{-1}_{f(p)}$ to

$T_{f(p)}S^{n+1}$ we have that the height function h (and hence the second fundamental form at p) of $f(M) \cap B(f(p), \varepsilon)$ is the zero function [1, 3]. Hence the sectional curvature K_p of $f(M)$ at $f(p)$ is 1. Similar argument shows that $K \equiv 1$ on $f(M)$. Finally, in [2], it is proved that $f(M)$ is a geodesic hypersphere and hence our result.

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