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SOME TRIPLED COINCIDENCE POINT THEOREMS FOR ALMOST GENERALIZED CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. In this paper, we prove tripled coincidence and common fixed point theorems for $F : X \times X \times X \to X$ and $g : X \to X$ satisfying almost generalized contractions in partially ordered metric spaces. The presented results generalize the theorem of Berinde and Borcut [*Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal.* **74** (15) (2011) 4889–4897]. Also, some examples are presented.

1. Introduction and preliminaries

Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations, partial differential equations, random differential equations and other related areas. Existence of fixed points in partially ordered metric spaces was investigated in 2004 by Ran and Reurings [29], and then by Nieto and Lopéz [28]. Further results in this direction were proved, see [5, 6, 17, 18, 19, 24, 25, 26, 27, 31].

The weak contraction principle was first introduced by Alber et al. [4] for Hilbert spaces and subsequently extended to metric spaces by Rhoades [30]. After that, fixed point problems involving weak contractions and mappings satisfying weak contraction type inequalities were considered in several works like [1, 8, 13, 15, 16, 22, 33].

Bhashkar and Lakshmikantham in [14] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Afterwards, Lakshmikantham and Ćirić [23] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered complete metric spaces. Various results on coupled fixed point have been obtained, since then see [3, 7, 9, 17, 20, 21, 24, 25].

On the other hand, Berinde and Borcut [12] introduced the concept of a tripled fixed point (see also the nice paper of Samet and Vetro [32]).

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Definition 1.1 ([12]). Let $(X,\leq)$ be a partially ordered set and $F : X\times X\times X \to X$. The mapping F is said to has the mixed monotone property if for any $x,y,z\in X$

$$x_1,\ x_2\in X,\quad x_1\leq x_2 \Longrightarrow F(x_1,y,z)\leq F(x_2,y,z),$$

$$y_1,\ y_2\in X,\quad y_1\leq y_2 \Longrightarrow F(x,y_1,z)\geq F(x,y_2,z),$$

$$z_1,\ z_2\in X,\quad z_1\leq z_2 \Longrightarrow F(x,y,z_1)\leq F(x,y,z_2).$$

Definition 1.2 ([12]). Let $F : X\times X\times X \to X$. An element (x,y,z) is called a tripled fixed point of F if

$$F(x,y,z)=x,\quad F(y,x,y)=y\quad \text{and}\ F(z,y,x)=z.$$

Also, Berinde and Borcut [12] proved the following theorem:

Theorem 1.1 ([12]). *Let $(X,\leq,d)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X\times X\times X\to X$ such that F has the mixed monotone property and there exist $j,r,l\geq 0$ with $j+r+l<1$ such that*

$$d(F(x,y,z),F(u,v,w))\leq jd(x,u)+rd(y,v)+ld(z,w), \tag{1}$$

for any $x,y,z,u,v,w\in X$ for which $x\leq u$, $v\leq y$ and $z\leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $a_n\to a$, then $a_n\leq a$ for all n,*
2. *if a non-increasing sequence $b_n\to b$, then $b\leq b_n$ for all n.*

If there exist $x_0,y_0,z_0\in X$ such that $x_0\leq F(x_0,y_0,z_0)$, $y_0\geq F(y_0,x_0,z_0)$ and $z_0\leq F(z_0,y_0,x_0)$, then there exist $x,y,z\in X$ such that

$$F(x,y,z)=x,\quad F(y,x,y)=y\quad \textit{and}\quad F(z,y,x)=z,$$

that is, F has a tripled fixed point.

Recently, Abbas, Aydi and Karapinar [2] introduced the following concepts.

Definition 1.3 ([2]). Let $(X,\leq)$ be a partially ordered set. Let $F : X\times X\times X\to X$ and $g : X\to X$. The mapping F is said to has the mixed g-monotone property if for any $x,y,z\in X$

$$x_1,\ x_2\in X,\quad gx_1\leq gx_2 \Longrightarrow F(x_1,y,z)\leq F(x_2,y,z),$$

$$y_1,\ y_2\in X,\quad gy_1\leq gy_2 \Longrightarrow F(x,y_1,z)\geq F(x,y_2,z),$$

$$z_1,\ z_2\in X,\quad gz_1\leq gz_2 \Longrightarrow F(x,y,z_1)\leq F(x,y,z_2).$$

Definition 1.4 ([2])**.** Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled coincidence point of F and g if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz.$$

(gx, gy, gz) is said a tripled point of coincidence of F and g.

Definition 1.5 ([2])**.** Let $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled common fixed point of F and g if

$$F(x, y, z) = gx = x, \quad F(y, x, y) = gy = y \quad \text{and} \quad F(z, y, x) = gz = z.$$

Definition 1.6 ([2])**.** Let X be a nonempty set. We say that the mappings $F : X \times X \times X \to X$ and $g : X \to X$ are commutative if for all $x, y, z \in X$

$$g(F(x, y, z)) = F(gx, gy, gz).$$

Now, let Φ be the set of all functions $\phi : [0,\infty) \to [0,\infty)$ such that

1. ϕ is non-decreasing,
2. $\phi(t) < t$ for all $t > 0$,
3. $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

From now on, we denote $X^3 = X \times X \times X$. For given mappings $F : X^3 \to X$ and $g : X \to X$, define

$$M(x, y, z, u, v, w) = \min\{d(F(x, y, z), gx), d(F(u, v, w), gx), d(F(u, v, w), gu)\}. \tag{2}$$

We say that such F and g verify almost generalized contractions if there exist $\phi \in \Phi$ and $L \geq 0$ such that

$$d(F(x, y, z), F(u, v, w)) \leq \phi\Big(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}\Big) + LM(x, y, z, u, v, w)$$

for any $x, y, z, u, v, w \in X$. Note that the concept of almost contractions were introduced by Berinde [10, 11].

In this paper, we establish tripled coincidence and common fixed point theorems for $F : X^3 \to X$ and $g : X \to X$ satisfying almost generalized contractions in partially ordered metric spaces. These results generalize Theorem 1.1 of Berinde and Borcut [12].

2. Main results

Our first result is given by the following:

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $L \geq 0$ such that*

$$d(F(x,y,z), F(u,v,w)) \leq \phi\Big(\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}\Big) + LM(x,y,z,u,v,w) \quad (3)$$

for any $x, y, z, u, v, w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose that $F(X^3) \subset g(X)$, g is continuous and commutes with F.
If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz,$$

that is, F and g have a tripled coincidence point.

Proof. Let $x_0, y_0, z_0 \in X$ be such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F(X^3) \subset g(X)$, then we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, x_0, y_0) \quad \text{and} \quad gz_1 = F(z_0, y_0, x_0). \quad (4)$$

Again, from $F(X^3) \subset g(X)$, continuing this process, we can construct sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, y_n) \quad \text{and} \quad gz_{n+1} = F(z_n, y_n, x_n). \quad (5)$$

We shall show that

$$gx_n \leq gx_{n+1}, \quad gy_{n+1} \leq gy_n \quad \text{and} \quad gz_n \leq gz_{n+1}, \text{ for } n = 0, 1, 2, \ldots \quad (6)$$

We shall use the mathematical induction. Since $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then by (4), we get

$$gx_0 \leq gx_1, \quad gy_1 \leq gy_0 \quad \text{and} \quad gz_0 \leq gz_1,$$

that is (6) holds for $n = 0$.
We presume that (6) holds for some $n \geq 1$. As F has the mixed g-monotone property and $gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$ and $gz_n \leq gz_{n+1}$, we obtain

$$\begin{aligned} gx_{n+1} = F(x_n, y_n, z_n) &\leq F(x_{n+1}, y_n, z_n) \\ &\leq F(x_{n+1}, y_n, z_{n+1}) \\ &\leq F(x_{n+1}, y_{n+1}, z_{n+1}) = gx_{n+2}, \end{aligned}$$

$$\begin{aligned} gy_{n+2} = F(y_{n+1}, x_{n+1}, y_{n+1}) &\le F(y_{n+1}, x_n, y_{n+1}) \\ &\le F(y_n, x_n, y_{n+1}) \\ &\le F(y_n, x_n, y_n) = gy_{n+1} \end{aligned}$$

and

$$\begin{aligned} gz_{n+1} = F(z_n, y_n, x_n) &\le F(z_{n+1}, y_n, x_n) \\ &\le F(z_{n+1}, y_{n+1}, x_n) \\ &\le F(z_{n+1}, y_{n+1}, x_{n+1}) = gz_{n+2}. \end{aligned}$$

Thus, (6) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1} \quad \text{and} \quad gz_n = gz_{n+1},$$

then, by (5), (x_n, y_n, z_n) is a tripled coincidence point of F and g. From now on, assume that at least for any $n \in \mathbb{N}$

$$gx_n \neq gx_{n+1} \quad \text{or} \quad gy_n \neq gy_{n+1} \quad \text{or} \quad gz_n \neq gz_{n+1}. \tag{7}$$

By (2) and (5), it is easy that

$$\begin{aligned} M(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n) &= M(y_n, x_n, y_n, y_{n-1}, x_{n-1}, y_{n-1}) \\ &= M(z_{n-1}, y_{n-1}, x_{n-1}, z_n, y_n, x_n) = 0. \end{aligned} \tag{8}$$

Due to (3) and (8), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ &\le \phi\Big(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}\Big) \\ &\quad + LM(x_{n-1}, y_{n-1}, z_{n-1}, x_n, y_n, z_n) \\ &= \phi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}), \end{aligned} \tag{9}$$

$$\begin{aligned} d(gy_{n+1}, gy_n) &= d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1})) \\ &\le \phi(\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) \\ &\quad + LM(y_n, x_n, y_n, y_{n-1}, x_{n-1}, y_{n-1}) \\ &= \phi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\ &\le \phi(\max\{d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}) \\ &\quad \text{since } \phi \text{ is non-decreasing} \end{aligned} \tag{10}$$

and

$$d(gz_n, gz_{n+1}) = d(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n))$$

$$\begin{aligned}&\leq \phi(\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\})\\&\quad +LM(z_{n-1},y_{n-1},x_{n-1},z_n,y_n,x_n)\\&=\phi(\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\}).\end{aligned}\tag{11}$$

Having in mind that $\phi(t)<t$ for all $t>0$, so from (9)-(11) we obtain that

$$\begin{aligned}0&<\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1})\}\\&\leq \phi(\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\})\\&<\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\}.\end{aligned}\tag{12}$$

It follows that

$$\begin{aligned}&\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1},d(gz_n,gz_{n+1})\}\\&\quad<\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\}.\end{aligned}$$

Thus, $\{\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1})\}\}$ is a positive decreasing sequence. Hence, there exists $r\geq 0$ such that

$$\lim_{n\to+\infty}\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1})\}=r.$$

Suppose that $r>0$. Letting $n\to+\infty$ in (12), we obtain that

$$0<r\leq \lim_{n\to+\infty}\phi(\max\{d(gz_{n-1},gz_n),d(gy_{n-1},gy_n),d(gx_{n-1},gx_n)\})=\lim_{t\to r^+}\phi(t)<r,\tag{13}$$

it is a contradiction. We deduce that

$$\lim_{n\to+\infty}\max\{d(gx_n,gx_{n+1}),d(gy_n,gy_{n+1}),d(gz_n,gz_{n+1})\}=0.\tag{14}$$

We shall show that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the metric space (X,d). Assume the contrary, that is, $\{gx_n\}$, $\{gy_n\}$ or $\{gz_n\}$ is not a Cauchy sequence, that is,

$$\lim_{n,m\to+\infty}d(gx_m,gx_n)\neq 0,\quad\text{or}\quad \lim_{n,m\to+\infty}d(gy_m,gy_n)\neq 0\quad\text{or}\quad \lim_{n,m\to+\infty}d(gz_m,gz_n)\neq 0.$$

This means that there exists $\varepsilon>0$ for which we can find subsequences of integers (m_k) and (n_k) with $n_k>m_k>k$ such that

$$\max\{d(gx_{m_k},gx_{n_k}),d(gy_{m_k},gy_{n_k}),d(gz_{m_k},gz_{n_k})\}\geq\varepsilon.\tag{15}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k>m_k$ and satisfying (15). Then

$$\max\{d(gx_{m_k}),gx_{n_k-1}),d(gy_{m_k},gy_{n_k-1}),d(gz_{m_k},gz_{n_k-1})\}<\varepsilon.\tag{16}$$

By triangular inequality and (16), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\le d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< \epsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Thus, by (14) we obtain

$$\lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k}) \le \lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k-1}) \le \varepsilon. \tag{17}$$

Similarly, we have

$$\lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k}) \le \lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k-1}) \le \varepsilon. \tag{18}$$

$$\lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k}) \le \lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k-1}) \le \varepsilon. \tag{19}$$

Again by (16), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\le d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &\le d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) \\ &\quad + d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) + \varepsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Letting $k \to +\infty$ and using (14), we get

$$\lim_{k\to+\infty} d(gx_{m_k}, gx_{n_k}) \le \lim_{k\to+\infty} d(gx_{m_k-1}, gx_{n_k-1}) \le \varepsilon, \tag{20}$$

$$\lim_{k\to+\infty} d(gy_{m_k}, gy_{n_k}) \le \lim_{k\to+\infty} d(gy_{m_k-1}, gy_{n_k-1}) \le \varepsilon, \tag{21}$$

and

$$\lim_{k\to+\infty} d(gz_{m_k}, gz_{n_k}) \le \lim_{k\to+\infty} d(gz_{m_k-1}, gz_{n_k-1}) \le \varepsilon. \tag{22}$$

Using (15) and (20)-(22), we have

$$\begin{aligned} &\lim_{k\to+\infty} \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\quad = \lim_{k\to+\infty} \max\{d(gx_{m_k-1}, gx_{n_k-1}), d(gy_{m_k-1}, gy_{n_k-1}), d(gz_{m_k-1}, gz_{n_k-1})\} \\ &\quad = \varepsilon. \end{aligned} \tag{23}$$

By (14), it is easy that

$$\begin{aligned} &\lim_{k\to+\infty} M(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, x_{n_k-1}, y_{n_k-1}, z_{n_k-1}) \\ &\quad = \lim_{k\to+\infty} M(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}, y_{m_k-1}, x_{m_k-1}, y_{m_k-1}) \\ &\quad = \lim_{k\to+\infty} M(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}, z_{n_k-1}, y_{n_k-1}, x_{n_k-1}) = 0. \end{aligned} \tag{24}$$

Now, by inequality (3) we obtain

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &= d(F(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}, z_{n_k-1})) \\ &\le \phi\Big(\max\{d(x_{m_k-1}, x_{n_k-1}), d(y_{m_k-1}, y_{n_k-1}), d(z_{m_k-1}, z_{n_k-1})\}\Big) \\ &\quad + LM(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, x_{n_k-1}, y_{n_k-1}, z_{n_k-1}), \end{aligned} \tag{25}$$

$$\begin{aligned} d(gy_{n_k}, gy_{m_k}) &= d(F(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}), F(y_{m_k-1}, x_{m_k-1}, y_{m_k-1})) \\ &\le \phi\Big(\max\{d(y_{m_k-1}, y_{n_k-1}), d(x_{m_k-1}, x_{n_k-1})\}\Big) \\ &\quad + LM(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}, y_{m_k-1}, x_{m_k-1}, y_{m_k-1}), \end{aligned} \tag{26}$$

and

$$\begin{aligned} d(gz_{m_k}, gz_{n_k}) &= d(F(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}), F(z_{n_k-1}, y_{n_k-1}, x_{n_k-1})) \\ &\le \phi\Big(\max\{d(x_{m_k-1}, x_{n_k-1}), d(y_{m_k-1}, y_{n_k-1}), d(z_{m_k-1}, z_{n_k-1})\}\Big) \\ &\quad + LM(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}, z_{n_k-1}, y_{n_k-1}, x_{n_k-1}). \end{aligned} \tag{27}$$

From (25)-(27), we deduce that

$$\begin{aligned} &\max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k})\} \\ &\quad \le \phi(\max\{d(x_{m_k-1}, x_{n_k-1}), d(y_{m_k-1}, y_{n_k-1}), d(z_{m_k-1}, z_{n_k-1})\}) \\ &\quad\quad + LM(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, z_{n_k-1}, y_{n_k-1}, x_{n_k-1}) \\ &\quad\quad + LM(y_{n_k-1}, x_{n_k-1}, y_{n_k-1}, y_{m_k-1}, x_{m_k-1}, y_{m_k-1}) \\ &\quad\quad + LM(z_{m_k-1}, y_{m_k-1}, x_{m_k-1}, z_{n_k-1}, y_{n_k-1}, x_{n_k-1}). \end{aligned} \tag{28}$$

Letting $k \to +\infty$ in (28) and having in mind (23) and (24), we get that

$$0 < \varepsilon \le \lim_{t \to \varepsilon^+} \phi(t) < \varepsilon,$$

it is a contradiction. Thus, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in (X, d). Since X is complete, there exist $x, y, z, u, v, w \in X$ such that

$$\lim_{n\to+\infty} gx_n = x, \quad \lim_{n\to+\infty} gy_n = y, \quad \text{and} \quad \lim_{n\to+\infty} gz_n = z. \tag{29}$$

From (29) and the continuity of g.

$$\lim_{n\to+\infty} g(gx_n) = gx, \quad \lim_{n\to+\infty} g(gy_n) = gy, \quad \text{and} \quad \lim_{n\to+\infty} g(gz_n) = gz. \tag{30}$$

By (5) and the commutativity of F and g, we have

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n), \tag{31}$$

$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n), \tag{32}$$

and

$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n). \tag{33}$$

Now we shall show that $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$.
Letting $n \to +\infty$ in (31)-(33), by (29), (30) and the continuity of F, we obtain

$$gx = \lim_{n\to+\infty} g(gx_{n+1}) = \lim_{n\to+\infty} F(gx_n, gy_n, gz_n) = F(\lim_{n\to+\infty} gx_n, \lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gz_n) = F(x, y, z),$$
$$gy = \lim_{n\to+\infty} g(gy_{n+1}) = \lim_{n\to+\infty} F(gy_n, gx_n, gy_n) = F(\lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gx_n, \lim_{n\to+\infty} gy_n) = F(y, x, y),$$

and

$$gz = \lim_{n\to+\infty} g(gz_{n+1}) = \lim_{n\to+\infty} F(gz_n, gy_n, gx_n) = F(\lim_{n\to+\infty} gz_n, \lim_{n\to+\infty} gy_n, \lim_{n\to+\infty} gx_n) = F(z, y, x).$$

We have proved that F and g have a tripled coincidence point. This completes the proof of Theorem 2.1.

In the next theorem, we omit the continuity hypothesis of F.

Theorem 2.2. *Let $(X, d, \leq)$ be a partially ordered metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Also, assume that there exist $\phi \in \Phi$ and $L \geq 0$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \phi\Big(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}\Big) + LM(x, y, z, u, v, w)$$

for any $x, y, z, u, v, w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose $F(X^3) \subset g(X)$, $(g(X), d)$ is a complete metric space and X has the following properties:

(i) *if non-decreasing sequence $a_n \to a$, then $a_n \leq a$ for all n,*

(ii) *if non-increasing sequence $b_n \to b$, then $b_n \geq b$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz,$$

that is, F and g have a tripled coincidence point.

Proof. Proceeding exactly as in Theorem 2.1, we have that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the complete metric space $(g(X), d)$. Then, there exist $x, y, z \in X$ such that

$$gx_n \to gx, \; gy_n \to gy \text{ and } gz_n \to gz. \tag{34}$$

Since $\{gx_n\}$, $\{gz_n\}$ are non-decreasing and $\{gy_n\}$ is non-increasing, then by the properties (i) and (ii) of X we have

$$gx_n \leq gx, \;\; gy_n \geq gy, \;\; gz_n \leq gz$$

for all n. If $gx_n = gx$, $gy_n = gy$ and $gz_n = gz$ for some $n \geq 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$, $gy \leq gy_{n+1} \leq gy_n = gy$, and $gz = gz_n \leq gz_{n+1} \leq gz = gz_n$. Therefore

$$gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \quad gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n),$$

that is, (x_n, y_n, z_n) is a tripled coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n) \neq (gx, gy, gz)$ for all $n \geq 0$. Consider now

$$\begin{aligned} d(gx, F(x,y,z)) &\leq d(gx, gx_{n+1}) + d(gx_{n+1}, F(x,y,z) \\ &= d(gx, gx_{n+1}) + d(F(x_n, y_n, z_n), F(x,y,z)) \\ &\leq d(gx, gx_{n+1}) + \phi(\max\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz)\}) + LM(x_n, y_n, z_n, x, y, z) \\ &< d(gx, gx_{n+1}) + \max\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz) + LM(x_n, y_n, z_n, x, y, z) \end{aligned} \tag{35}$$

where

$$M(x_n, y_n, z_n, x, y, z) = \min\Big\{ d(F(x_n, y_n, z_n), gx_n), d(F(x_n, y_n, z_n), gx), d(F(x,y,z), gx) \Big\}.$$

Taking $n \to \infty$ and using (34), the quantity $M(x_n, y_n, z_n, x, y, z)$ tends to 0, also the right-hand side of (35) tends to 0, so we get that $d(gx, F(x,y,z)) = 0$. Thus, $gx = F(x,y,z)$. Analogously, we get that

$$F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz.$$

Thus, we proved that F and g have a tripled coincidence point. This completes the proof of Theorem 2.2.

Corollary 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume that there exist $\phi \in \Phi$ and $L \geq 0$ such that*

$$d(F(x,y,z), F(u,v,w)) \leq \phi\left(\frac{d(gx,gu) + d(gy,gv) + d(gz,gw)}{3}\right) + LM(x,y,z,u,v,w),$$

for any $x, y, z, u, v, w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose $F(X^3) \subset g(X)$, g is continuous and commutes with F.
If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \textit{and} \quad F(z,y,x) = gz.$$

Proof. It suffices to remark that

$$\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3} \leq \max\{d(gx,gu),d(gy,gv),d(gz,gw)\}.$$

Then, we apply Theorem 2.1, since ϕ is non-decreasing.

Similarly, we have from Theorem 2.2 the following corollary.

Corollary 2.2. *Let $(X,d,\leq)$ be a partially ordered metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Assume that there exist $\phi \in \Phi$ and $L \geq 0$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq \phi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}\right) + LM(x,y,z,u,v,w),$$

for any $x,y,z,u,v,w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose that $F(X^3) \subset g(X)$, $(g(X),d)$ is a complete metric space and X has the following properties:

(i) *if non-decreasing sequence $a_n \to a$, then $a_n \leq a$ for all n,*

(ii) *if non-increasing sequence $b_n \to b$, then $b_n \geq b$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0,y_0,z_0)$, $gy_0 \geq F(y_0,x_0,y_0)$ and $gz_0 \leq F(z_0,y_0,x_0)$, then there exist $x,y,z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz.$$

Corollary 2.3. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X^3 \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume that there exist $k \in [0,1)$ and $L \geq 0$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq k\max\{d(gx,gu),d(gy,gv),d(gz,gw)\} + LM(x,y,z,u,v,w),$$

for any $x,y,z,u,v,w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose that $F(X^3) \subset g(X)$, g is continuous and commutes with F.
If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0,y_0,z_0)$, $gy_0 \geq F(y_0,x_0,y_0)$ and $gz_0 \leq F(z_0,y_0,x_0)$, then there exist $x,y,z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz.$$

Proof. It suffices to take $\phi(t) = kt$ in Theorem 2.1.

Corollary 2.4. *Let $(X,d,\leq)$ be a partially ordered set metric space. Suppose $F: X^3 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property. Assume that there exist $k \in [0,1)$ and $L \geq 0$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq k \max\{d(gx,gu),d(gy,gv),d(gz,gw)\} + LM(x,y,z,u,v,w),$$

for any $x,y,z,u,v,w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose $F(X^3) \subset g(X)$, $(g(X),d)$ is a complete metric space and X has the following properties:

(i) *if non-decreasing sequence $a_n \to a$, then $a_n \leq a$ for all n,*

(ii) *if non-increasing sequence $b_n \to b$, then $b_n \geq b$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0,y_0,z_0)$, $gy_0 \geq F(y_0,x_0,y_0)$ and $gz_0 \leq F(z_0,y_0,x_0)$, then there exist $x,y,z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz.$$

Proof. It suffices to take $\phi(t) = kt$ in Theorem 2.2.

Corollary 2.5. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F: X^3 \to X$ and $g: X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume that there exist $k \in [0,1)$ and $L \geq 0$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq \frac{k}{3}(d(gx,gu)+d(gy,gv)+d(gz,gw)) + LM(x,y,z,u,v,w),$$

for any $x,y,z,u,v,w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose $F(X^3) \subset g(X)$, g is continuous and commutes with F.
If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0,y_0,z_0)$, $gy_0 \geq F(y_0,x_0,y_0)$ and $gz_0 \leq F(z_0,y_0,x_0)$, then there exist $x,y,z \in X$ such that

$$F(x,y,z) = gx, \quad F(y,x,y) = gy \quad \text{and} \quad F(z,y,x) = gz.$$

Proof. It suffices to take $\phi(t) = kt$ in Corollary 2.1.

Corollary 2.6. *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F: X^3 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property. Assume that there exist $k \in [0,1)$ and $L \geq 0$ such that*

$$d(F(x,y,z),F(u,v,w)) \leq \frac{k}{3}(d(gx,gu)+d(gy,gv)+d(gz,gw)) + LM(x,y,z,u,v,w),$$

for any $x,y,z,u,v,w \in X$ for which $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Also, suppose that $F(X^3) \subset g(X)$, $(g(X),d)$ is a complete metric space and X has the following properties:

(i) *if non-decreasing sequence* $a_n \to a$, *then* $a_n \le a$ *for all* n,

(ii) *if non-increasing sequence* $b_n \to b$, *then* $b_n \ge b$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that* $gx_0 \le F(x_0, y_0, z_0)$, $gy_0 \ge F(y_0, x_0, y_0)$ *and* $gz_0 \le F(z_0, y_0, x_0)$, *then there exist* $x, y, z \in X$ *such that*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \textit{and} \quad F(z, y, x) = gz.$$

Proof. It follows by taking $\phi(t) = kt$ in Corollary 2.2.

Remark 1.
- Taking $L = 0$, $g = Id_X$, the identity on X and $\phi(t) = kt$, $k \in [0, 1)$ in Corollary 2.5, we get Theorem 7 of Berinde and Borcut [12] (with $j = l = r = \frac{k}{3}$).
- Taking $L = 0$, $g = Id_X$ and $\phi(t) = kt$, $k \in [0, 1)$ in Corollary 2.6, we get Theorem 8 of Berinde and Borcut [12] (with $j = l = r = \frac{k}{3}$).
- Corollary 2.3 generalizes Theorem 7 of Berinde and Borcut [12].
- Corollary 2.4 generalizes Theorem 8 of Berinde and Borcut [12]
- Corollary 2.5 and Corollary 2.6 are the analogous of Theorem 2.1 and Theorem 2.2 of Lakshmikantham and Ćirić [23] for coupled fixed point results by taking $L = 0$.

Now, we shall prove the existence and uniqueness of tripled common fixed point. For a product X^3 of a partial ordered set $(X, \le)$, we define a partial ordering in the following way: For all (x, y, z) and (u, v, r) in X^3

$$(x, y, z) \le (u, v, r) \Leftrightarrow x \le u, \quad y \ge v, \quad z \le r. \tag{36}$$

We say that (x, y, z) and (u, v, w) are comparable if

$$(x, y, z) \le (u, v, r) \quad \text{or} \quad (u, v, r) \le (x, y, z).$$

Also, we say that (x, y, z) is equal to (u, v, r) if and only if $x = u$, $y = v$, $z = r$.

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1, suppose that that for all* (x, y, z), $(u, v, r) \in X^3$, *there exists* $(a, b, c) \in X \times X \times X$ *such that* $(F(a, b, c), F(b, a, b), F(c, b, a))$ *is comparable to* $(F(x, y, z), F(y, x, y), F(z, y, x))$ *and* $(F(u, v, r), F(v, u, v), F(r, v, u))$. *Then,* F *and* g *have a unique tripled common fixed point* (x, y, z) *such that*

$$x = gx = F(x, y, z), \quad x = gx = F(x, y, z) \quad \textit{and} \quad z = gz = F(z, y, x).$$

Proof. The set of tripled coincidence points of F and g is not empty due to Theorem 2.1. Assume now, (x,y,z) and (u,v,r) are two tripled coincidence points of F and g, that is,

$$\begin{aligned} F(x,y,z) &= gx, \quad F(u,v,r) = gu,\\ F(y,x,y) &= gy, \quad F(v,u,v) = gv,\\ F(z,y,x) &= gz, \quad F(r,v,u) = gr. \end{aligned} \tag{37}$$

We shall show that (gx,gy,gz) and (gu,gv,gr) are equal. By assumption, there exists $(a,b,c) \in X^3$ such that $(F(a,b,c),F(b,a,b),F(c,b,a))$ is comparable to $(F(x,y,z),F(y,x,y),F(z,y,x))$ and $(F(u,v,r),F(v,u,v),F(r,v,u))$.

Define sequences $\{ga_n\},\{gb_n\}$ and $\{gc_n\}$ such that

$$a_0 = a, \quad b_0 = b, \quad c_0 = c, \quad \text{and for any } \ n \ge 1$$

$$\begin{aligned} ga_n &= F(a_{n-1},b_{n-1},c_{n-1}),\\ gb_n &= F(b_{n-1},a_{n-1},b_{n-1}),\\ gc_n &= F(c_{n-1},b_{n-1},a_{n-1}), \end{aligned} \tag{38}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, and on the same way define the sequences $\{gx_n\},\{gy_n\}$, $\{gz_n\}$ and $\{gu_n\},\{gv_n\}$, $\{gr_n\}$. Then, it is easy that

$$\begin{aligned} gx_n &= F(x,y,z), \quad gu_n = F(u,v,r),\\ gy_n &= F(y,x,y,), \quad gv_n = F(v,u,v),\\ gz_n &= F(z,y,x), \quad gr_n = F(r,v,u), \end{aligned} \tag{39}$$

for all $n \ge 1$. Since $(F(x,y,z),F(y,x,y),F(z,y,x)) = (gx_1,gy_1,gz_1) = (gx,gy,gz)$ is comparable to $(F(a,b,c),F(b,a,b),F(c,b,a)) = (ga_1,gb_1,gc_1)$, then it is easy to show $(gx,gy,gz) \ge (ga_1,gb_1,gc_1)$. Recursively, we get that

$$(gx,gy,gz) \ge (ga_n,gb_n,gc_n) \quad \text{for all } n. \tag{40}$$

By (2) and (37), it is easy that

$$M(a_n,b_n,c_n,x,y,z) = M(y,x,y,b_n,a_n,b_n) = M(c_n,b_n,a_n,z,y,x) = 0 \tag{41}$$

for all $n \ge 0$. By (3), (40) and (41), we have

$$\begin{aligned} d(ga_{n+1},gx) &= d(F(a_n,b_n,c_n),F(x,y,z))\\ &\le \phi(\max\{d(gx,ga_n),d(gy,gb_n),d(gz,gc_n)\}) + LM(a_n,b_n,c_n,x,y,z)\\ &= \phi(\max\{d(gx,ga_n),d(gy,gb_n),d(gz,gc_n)\}), \end{aligned} \tag{42}$$

$$\begin{aligned} d(gy, gb_{n+1}) &= d(F(F(y,x,y), b_n, a_n, b_n)) \\ &\le \phi(\max\{d(ga_n, gx), d(gb_n, gy)\}) + LM(y,x,y,b_n,a_n,b_n) \\ &= \phi(\max\{d(gb_n, gy), d(ga_n, gx)\}) \\ &\le \phi(\max\{d(gb_n, gy), d(ga_n, gx), d(gc_n, gz)\}), \end{aligned} \tag{43}$$

and

$$\begin{aligned} d(gz, gc_{n+1}) &= d(F(c_n, b_n, a_n), F(z,y,x)) \\ &\le \phi(\max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n)\}) + LM(z,y,x,c_n,b_n,a_n) \\ &= \phi(\max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n)\}). \end{aligned} \tag{44}$$

From (42)-(44), it follows that

$$\max\{d(gz, gc_{n+1}), d(gy, gb_{n+1}), d(gx, ga_{n+1})\} \le \phi(\max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n)\}).$$

Therefore, for each $n \ge 1$,

$$\max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n)\} \le \phi^n(\max\{d(gz, gc_0), d(gy, gb_0), d(gx, ga_0)\}). \tag{45}$$

It is known that $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ imply $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$. Thus, from (45),

$$\lim_{n\to\infty} \max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n)\} = 0.$$

This yields that

$$\lim_{n\to\infty} d(gx, ga_n) = \lim_{n\to\infty} d(gy, gb_n) = \lim_{n\to\infty} d(gz, gc_n) = 0. \tag{46}$$

Analogously, we show that

$$\lim_{n\to\infty} d(gu, ga_n) = \lim_{n\to\infty} d(gv, gb_n) = \lim_{n\to\infty} d(gr, gc_n) = 0. \tag{47}$$

Combining (46) and (47) yields that (gx, gy, gz) and (gu, gv, gr) are equal.

Since $gx = F(x,y,z)$, $gy = F(y,x,y)$ and $gz = F(z,y,x)$, by commutativity of F and g, we have

$$\begin{aligned} gx' &= g(gx) = g(F(x,y,z)) = F(gx, gy, gz), \\ gy' &= g(gy) = g(F(y,x,y)) = F(gy, gx, gy), \end{aligned}$$

and

$$gz' = g(gz) = g(F(z,y,x)) = F(gz, gy, gx),$$

where $gx = x'$, $gy = y'$ and $gz = z'$. Thus, (x', y', z') is a tripled coincidence point of F and g. Consequently, (gx', gy', gz') and (gx, gy, gz) are equal. We deduce

$$gx' = gx = x', \quad gy' = gy = y' \quad \text{and} \quad gz' = gz = z'.$$

Therefore, (x', y', z') is a tripled common fixed of F and g. Its uniqueness follows easily from (3).

We present the following examples illustrating our results.

Example 2.1. Let $X = \mathbb{R}$ be endowed with the Euclidian metric $d(x, y) = |x - y|$, for all $x, y \in X$ and be ordered by the following relation

$$x \leq_X y \Longleftrightarrow x = y \text{ or } (x, y \in [0,1] \text{ and } x \leq y),$$

where $\leq$ be the usual ordering. Let $g : X \to X$ and $F : X^3 \to X$ be defined by

$$g(x) = \begin{cases} \frac{1}{10}x & \text{if } x < 0 \\ x & \text{if } x \in [0,1] \\ \frac{1}{10}x + \frac{9}{10} & \text{if } x > 1, \end{cases} \qquad F(x, y, z) = \frac{x - y + z}{4}.$$

It is obvious that $F(X^3) \subset g(X)$, F has the mixed g-monotone property and $(g(X), d)$ is a complete metric space.

Take $L \geq 0$ arbitrary and $\phi : [0, \infty) \to [0, \infty)$ be given by $\phi(t) = \frac{3}{4}t$ for all $t \in [0, \infty)$. We will check that condition (3) is fulfilled for all $x, y, z, u, v, w \in X$ satisfying $gx \leq_X gu$, $gv \leq_X gy$ and $gz \leq_X gw$. The following cases are possible:

- Case 1. $x, u, y, v, z, w \in [0,1]$. In this case

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) &= \frac{u - x}{4} + \frac{y - v}{4} + \frac{w - z}{4} \\ &\leq \frac{3}{4}\max\{d(gx, gu), d(gy, gv), d(gz, gw)\} \\ &\leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) + LM(x, y, z, u, v, w). \end{aligned}$$

- Case 2. $x, u, y, v \in [0,1]$ et $z, w \notin [0,1]$. Here, $gz, gw \notin [0,1]$ and since they must be comparable, $gz = gw$ and $z = w$. In this case

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) &= \frac{u - x}{4} + \frac{y - v}{4} \\ &\leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) + LM(x, y, z, u, v, w). \end{aligned}$$

- Case 3. The cases where ($x, u, z, w \in [0,1]$ and $y, v \notin [0,1]$) or ($y, v, z, w \in [0,1]$ and $x, u \notin [0,1]$) are treated analogously to Case 2.
- Case 4. $x, u \in [0,1]$ et $y, v, z, w \notin [0,1]$. Here, $gy, gv, gz, gw \notin [0,1]$ and since they must be comparable, $gy = gv$ and $gz = gw$, so $y = v$ and $z = w$. In this case

$$\begin{aligned} d(F(x, y, z), F(u, v, w)) &= \frac{u - x}{4} \\ &\leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) + LM(x, y, z, u, v, w). \end{aligned}$$

- Case 5. The cases where ($y, v \in [0,1]$ et $x, u, z, w \notin [0,1]$) or ($z, w \in [0,1]$ et $x, u, y, v \notin [0,1]$) are treated analogously to Case 4.

- Case 6. $x, u, y, v, z, w \notin [0,1]$. Then the only possibility for gx, gu, as well as gy, gv and gz, gw to be comparable is that $x = u$, $y = v$ and $z = w$. In this case condition (3) is trivially satisfied.

We conclude that all the conditions of Theorem 2.2 are satisfied. The mappings g and F have a tripled coincidence (common) fixed point $(0,0,0)$.

Example 2.2. Let $X = [0,\infty)$ with the Euclidian metric and the following order relation:

$$x, y \in X,\ x \leq_X y \Leftrightarrow x = y = 0 \text{ or } (x, y \in (0,\infty) \text{ and } x \leq y),$$

where $\leq$ be the usual ordering.
Let $g : X \to X$ and $F : X^3 \to X$ be given by $gx = \frac{2}{3}x$ and

$$F(x,y,z) = \begin{cases} 1, & if\ xyz \neq 0 \\ 0, & if\ xyz = 0 \end{cases}$$

for all $x, y, z, u, v, w \in X$.
Let $\phi : [0,\infty) \to [0,\infty)$ be defined $\phi(t) = \frac{3t}{4}$ for all $t \in [0,\infty)$.
It is easy to check that all the conditions of Theorem 2.2 are satisfied (for all $L \geq 0$). Applying Theorem 2.2 we conclude that F and g have a tripled coincidence point, which is $(0,0,0)$.

Example 2.3. Let $X = \mathbb{R}$ be endowed with the usual order $\leq$ and the Euclidian metric. Define mappings $g : X \to X$ and $F : X^3 \to X$ by

$$gx = 7x - 1 \quad \text{and } F(x,y,z) = 2x - 2y + 2z + 1.$$

Obviously, $F(X^3) \subset g(X)$, F has the mixed g-monotone property and $(g(X), d)$ is a complete metric space.

Take $L \geq 0$ arbitrary and $\phi : [0,\infty) \to [0,\infty)$ be given by $\phi(t) = \frac{6}{7}t$ for all $t \in [0,\infty)$. We will check that condition (3) is fulfilled for all $x, y, z, u, v, w \in X$ satisfying $gx \leq gu$, $gv \leq gy$ and $gz \leq gw$. Indeed

$$\begin{aligned} d(F(x,y,z), F(u,v,w)) &= 2(u-x) + 2(y-v) + 2(w-z) \\ &\leq \phi(\max\{7(u-x), 7(y-v), 7(w-z)\}) \\ &\leq \phi(\max\{d(gx,gu), d(gy,gv), d(gz,gw)\}) + LM(x,y,z,u,v,w). \end{aligned}$$

It is clear all conditions of Theorem 2.2 are satisfied (it suffices to take $x_0 = z_0 = 0$ and $y_0 = \frac{2}{3}$). So F and g have a tripled coincidence point. Here, $(\frac{2}{5}, \frac{2}{5}, \frac{2}{5})$ is a tripled coincidence point of F and g.

Note that Theorem 1.1 (main result of Berinde and Borcut) is not applicable in this case. Indeed, for $x, y, z, u, v, w \in X$ with $x < u$, $v = y$ and $z = w$, we have

$$d(F(x,y,z), F(u,v,w)) = 2(u-x) > jd(x,u) = jd(x,u) + rd(y,v) + ld(z,w),$$

for all $j, r, l \geq 0$ such that $j + r + l < 1$.

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